12. *Stochastic Flows of Automorphisms of G-structures of Degree r*

By Hiroshi AKIYAMA
Department of Applied Mathematics, Faculty of Engineering, Shizuoka University

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1. Introduction. Let M be a σ-compact connected C^∞ manifold of dimension n, and let $P^r(M)$ be the bundle of frames of r-th order contact over M with structure group $G^r(n)$ and natural projection π ([6], [7]). The purpose of this paper is to give a condition that a stochastic flow of diffeomorphisms generated by a stochastic differential equation on M be a stochastic flow of automorphisms of a G-structure of degree r (i.e. a G-subbundle of $P^r(M)$) for a closed subgroup G of $G^r(n)$ by using Itô's formula for fields of geometric objects ([1]). Our main result (Theorem 3.1) generalizes some results in [4], [9], [1] on stochastic flows of diffeomorphisms leaving tensor fields invariant.

We assume all non-probabilistic maps (and vector fields) in this paper are smooth. The tangent bundle over M is denoted by $T(M)$, and the tangent space at $x \in M$ by $T_x(M)$. Throughout the paper, indices take the following values: α, $\beta = 1, 2, \cdots, k$; $\lambda = 0, 1, \cdots, k$.

2. Preliminaries. Let M and G be as above. The quotient space $P^r(M)/G$ is then a fiber bundle with standard fiber $G^r(n)/G$ associated with $P^r(M)$. There is a natural one-to-one correspondence between the sections $M \to P^r(M)/G$ and the G-structures of degree r on M ([8, pp. 57–58]). A transformation φ of M induces a transformation $\tilde{\varphi}$ of $P^r(M)$ by $\tilde{\varphi}(j_0^r(f)) = j_0^r(\varphi \circ f)$ for any $j_0^r(f) \in P^r(M)$, where $j_0^r(f)$ is the r-jet at the origin $0 \in \boldsymbol{R}^n$ given by a diffeomorphism f of an open neighborhood of $0 \in \boldsymbol{R}^n$ onto an open set of M with $\pi(j_0^r(f)) := f(0)$. Then $\tilde{\varphi}$ induces a transformation $\bar{\varphi}$ of $P^r(M)/G$ such that $\bar{\varphi} \circ \mu = \mu \circ \tilde{\varphi}$, where $\mu: P^r(M) \to P^r(M)/G$ is the projection. For a section $\sigma: M \to P^r(M)/G$, we define a section $\varphi^\# \sigma: M \to P^r(M)/G$ by $\varphi^\# \sigma = \bar{\varphi}^{-1} \circ \sigma \circ \varphi$.

Correspondingly, a vector field $X: x \mapsto X(x) \in T_x(M)$, $x \in M$, on M induces a vector field $\tilde{X}$ on $P^r(M)$ and a vector field $\bar{X}$ on $P^r(M)/G$ in a natural manner, since X generates a local one-parameter group of local transformations φ_t of M and φ_t induces naturally a local one-parameter group of local transformations $\tilde{\varphi}_t$ [resp. $\bar{\varphi}_t$] of $P^r(M)$ [resp. $P^r(M)/G$]. We set $\varphi_t^\# \sigma = (\bar{\varphi}_t)^{-1} \circ \sigma \circ \varphi_t$. The vector field $\tilde{X}$ [resp. $\bar{X}$] is called the *natural lift of* X *to* $P^r(M)$ [resp. $P^r(M)/G$]. We denote by $\hat{L}_X \sigma: M \to T(P^r(M)/G)$ the *Lie derivative of* σ *with respect to* X *in the sense of Salvioli* ([10]); it is defined by

$$(\hat{L}_X\sigma)(x) := \frac{d}{dt}(\varphi_t^{\#}\sigma)(x)|_{t=0}$$
$$=\sigma_*(X(x))-\bar{X}(\sigma(x)) \in T_{\sigma(x)}(P^r(M)/G), \qquad x \in M,$$

where σ_* stands for the differential of σ.

Lemma 2.1. *Let G be a closed subgroup of $G^r(n)$. Let P be a G-structure of degree r on M, and let $\sigma: M \to P^r(M)/G$ be the corresponding section. Then:*

(1) *For a transformation φ of M, the G-structure of degree r corresponding to $\varphi^{\#}\sigma$ is given by $\tilde{\varphi}^{-1}(P)$.*

(2) *A transformation φ of M is an automorphism of P if and only if $\varphi^{\#}\sigma=\sigma$.*

(3) *A vector field X on M is an infinitesimal automorphism of P if and only if $\hat{L}_X\sigma=0$.*

Proof. (1) Since $\mu=(\varphi^{\#}\sigma)\circ\pi$ holds if and only if $\mu\circ\tilde{\varphi}=\bar{\varphi}\circ\mu=\sigma\circ\varphi\circ\pi=\sigma\circ\pi\circ\tilde{\varphi}$, we have $\{p \in P^r(M): \mu(p)=\varphi^{\#}\sigma(\pi(p))\}=\tilde{\varphi}^{-1}(P)$. (2) and (3) follow from (1).

3. Main result. Let M and G be as before. Let $X_0, X_1, \cdots, X_k$ be vector fields on M. For each λ, let $\tilde{X}_\lambda$ denote the natural lift of X_λ to $P^r(M)$. Consider the following stochastic differential equation in the Stratonovich form:

$$dp_t=\sum_\lambda \tilde{X}_\lambda(p_t)\circ dw_t^\lambda, \tag{3.1}$$

where $w_t^0\equiv t$, and $w_t=(w_t^1, \cdots, w_t^k)$ is a k-dimensional Wiener process canonically realized on the k-dimensional standard Wiener space (cf. [5], [9]). The solution with initial condition $p_s=p \in P^r(M)$ is denoted by $p_{s,t}(p)=(p_{s,t}(p, w))$, so that $p_{s,t}$ is a (stochastic) map $p_{s,t}: P^r(M)\to P^r(M)$.

Now we state our main result.

Theorem 3.1. *For a closed subgroup G of $G^r(n)$, suppose a G-structure P of degree r is given on M. Let $\sigma: M\to P^r(M)/G$ be the section corresponding to P* (cf. §2). *Assume* (3.1) *generates a stochastic flow of diffeomorphisms $p_{s,t}$ $(0\le s\le t)$ of $P^r(M)$, a.s. Then the stochastic flow of diffeomorphisms $\xi_{s,t}$ of M induced by $p_{s,t}$ is a stochastic flow of automorphisms of P, a.s. if and only if $\hat{L}_{X_\lambda}\sigma=0$ for each λ.*

Before proving Theorem 3.1, we note that $\xi_{s,t}$ is also generated by the stochastic differential equation

$$d\xi_t=\sum_\lambda X_\lambda(\xi_t)\circ dw_t^\lambda. \tag{3.2}$$

Moreover, for almost all w, we have $p_{s,t}(j_0^r(f))=j_0^r(\xi_{s,t}\circ f)=\tilde{\xi}_{s,t}(j_0^r(f))$ for any $j_0^r(f) \in P^r(M)$ and s, t with $0\le s\le t$.

Proof of Theorem 3.1. As is easily seen, $p_{s,t}$ induces a stochastic flow of diffeomorphisms $\eta_{s,t}(=\bar{\xi}_{s,t})$ of $P^r(M)/G$ in a natural way; $\eta_{s,t}$ is also generated by the stochastic differential equation

$$d\eta_t=\sum_\lambda \bar{X}_\lambda(\eta_t)\circ dw_t^\lambda,$$

where each $\bar{X}_\lambda$ is the natural lift of X_λ to $P^r(M)/G$. Consider the *stochastic deformation* $\xi_{s,t}^{\#}\sigma=\eta_{s,t}^{-1}\circ\sigma\circ\xi_{s,t}$ of σ by $\xi_{s,t}^{-1}$. By Lemma 2.1, we have only to prove that $\xi^{\#}_{s,t}\sigma=\sigma$ (a.s.) holds if and only if $\hat{L}_{X_\lambda}\sigma=0$ for each λ. To prove this, we shall use the following theorem, in which the following notation is used: If Y is either a tangent vector or a vector field on a manifold N, we denote by $Y[H]$ the operation of Y on a function $H: N\to \boldsymbol{R}$.

Theorem 3.2 (Itô's formula for $\xi^{\#}_{s,t}\sigma$). *For every function $F: P^r(M)/G \to \boldsymbol{R}$, it holds that*

$$\begin{aligned} F\circ(\xi^{\#}_{s,t}\sigma)(x)-F\circ\sigma(x) &= \sum_\lambda \Phi^\lambda_{s,t}(x,F)+\frac{1}{2}\sum_\alpha\int_s^t (X_\alpha(\xi_{s,u}(x))[((\hat{L}_{X_\alpha}\sigma)(\cdot))[F\circ\eta_{s,u}^{-1}]] \\ &\quad -((\hat{L}_{X_\alpha}\sigma)\circ\xi_{s,u}(x))[\bar{X}_\alpha[F\circ\eta_{s,u}^{-1}]])\cdot du, \qquad (x\in M), \end{aligned}$$

where $\cdot dw^\lambda_u$ stands for the Itô stochastic differential, and

$$\Phi^\lambda_{s,t}(x,F) := \int_s^t (\eta_{s,u})_*^{-1}((\hat{L}_{X_\lambda}\sigma)\circ\xi_{s,u}(x))[F]\cdot dw^\lambda_u.$$

Proof. Use [1, Theorem 3.1 (cf. § 5.2)].

Now that Theorem 3.2 is available, we can finish our proof of Theorem 3.1 easily: By Theorem 3.2, we see that $\xi^{\#}_{s,t}\sigma=\sigma$ if $\hat{L}_{X_\lambda}\sigma=0$ for each λ. Conversely, suppose $\xi^{\#}_{s,t}\sigma=\sigma$. Then, by Theorem 3.2, the (local) martingale part of the continuous semimartingale $F\circ(\xi^{\#}_{s,t}\sigma)(x)-F\circ\sigma(x)$ $(=0)$ is given by $\sum_\alpha \Phi^\alpha_{s,t}(x,F)$, and the bounded variation part of $(\sum_\beta\Phi^\beta_{s,t}(x,F))\cdot\Phi^\alpha_{s,t}(x,F)$ is

$$\int_s^t \{(\eta_{s,u})_*^{-1}((\hat{L}_{X_\alpha}\sigma)\circ\xi_{s,u}(x))[F]\}^2\cdot du=0. \tag{3.3}$$

The integrand of (3.3) being continuous in u, it holds that $(\eta_{s,u})_*^{-1}((\hat{L}_{X_\alpha}\sigma)\circ \xi_{s,u}(x))[F]=0$ for any u $(\geq s)$, so that $(\hat{L}_{X_\alpha}\sigma)(\cdot)[F]=0$ and thus $\hat{L}_{X_\alpha}\sigma=0$ since F is arbitrary. Using Theorem 3.2 again, we have moreover $\hat{L}_{X_0}\sigma=0$. (The "only if" part of Theorem 3.1 can also be proved by using a backward Itô's formula in [3].)

For example, let P be a projective structure on M with $n\geq 2$, $r=2$ and $G=H^2(n)$ in the sense of [7] (cf. [6]). Then, with respect to P, $\xi_{s,t}$ generated by (3.2) is a stochastic flow of projective transformations of M if and only if each X_λ is an infinitesimal projective transformation (see also [1], [2]).

References

[1] H. Akiyama: On Itô's formula for certain fields of geometric objects. J. Math. Soc. Japan, **39**, 79–91 (1987).

[2] ——: Applications of nonstandard analysis to stochastic flows and heat kernels on manifolds. Geometry of Manifolds (ed. K. Shiohama). Academic Press, pp. 3–27 (1989).

[3] ——: Backward Itô's formula for sections of a fibered manifold. J. Math. Soc. Japan, **42**, 327–340 (1990).

[4] J.-M. Bismut: Mécanique Aléatoire. Lect. Notes in Math., **866**, Springer (1981).

[5] N. Ikeda and S. Watanabe: Stochastic Differential Equations and Diffusion Processes. 2nd ed., North-Holland/Kodansha (1989).

[6] S. Kobayashi: Transformation Groups in Differential Geometry. Springer (1972).
[7] S. Kobayashi and T. Nagano: On projective connections. J. Math. Mech., **13**, 215–235 (1964).
[8] S. Kobayashi and K. Nomizu: Foundations of Differential Geometry, I. Wiley (Interscience) (1963).
[9] H. Kunita: Stochastic differential equations and stochastic flows of diffeomorphisms. École d'Été de Probab. de Saint Flour XII-1982, Lect. Notes in Math., **1097**, Springer, pp. 143–303 (1984).
[10] S. Salvioli: On the theory of geometric objects. J. Differential Geometry, **7**, 257–278 (1972).