11. On Automorphism Groups of Compact Riemann Surfaces with Prescribed Group Structure

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Let X be a compact Riemann surface of genus $g \geq 2$ and let $\mathrm{Aut}(X)$ be the group of all conformal automorphisms on X. Let $\rho: \mathrm{Aut}(X) \to GL(g, \boldsymbol{C})$ denote the canonical representation for a (fixed) basis $\{\xi_1, \cdots, \xi_g\}$ of abelian differentials of the first kind on X. In fact, for a $\sigma \in \mathrm{Aut}(X)$, we define the matrix $(s_{ij}) \in GL(g, \boldsymbol{C})$ by the relation:

$$\sigma^*(\xi_i) = \sum_{j=1}^{g} s_{ij}\xi_j \qquad (i=1, \cdots, g).$$

Here $\sigma^*(\xi_i)$ means the pull-back of ξ_i by σ. We denote by $\rho(AG; X)$ the image of a subgroup AG of $\mathrm{Aut}(X)$ by ρ. The purpose of this paper is to investigate conditions for a non abelian subgroup of $GL(g, \boldsymbol{C})$ of order 8 to be conjugate to some $\rho(AG; X)$ (for some AG and some X). We say that $G \subset GL(g, \boldsymbol{C})$ arises from a compact Riemann surface of genus g if G has the above property.

A more detailed account will be published elsewhere.

§1. Preliminaries. Let G be a finite subgroup of $GL(g, \boldsymbol{C})$ and H a non-trivial cyclic subgroup of G. Define two sets $CY(G)$ and $CY(G; H)$ by

$CY(G) := \{K; K \text{ is a non-trivial cyclic subgroup of } G\}$,

$CY(G; H) := \{K \in CY(G); K \text{ contains } H \text{ strictly}\}$.

Definition (see [1]). We say that G satisfies the *CY-condition*, if any element of $CY(G)$ is $GL(g, \boldsymbol{C})$-conjugate to a group arising from Riemann surfaces of genus g.

Definition. We say that G satisfies *E condition* if for every element A of G, $\mathrm{Tr}(A) + \mathrm{Tr}(A^{-1})$ is an integer. Further we define as follows:

$r(H) := 2 - (\mathrm{Tr}(A) + \mathrm{Tr}(A^{-1}))$, where $H = \langle A \rangle$.

$r_*(H; G) = r(H) - \sum_K r_*(K; G)$, where K ranges over the set $CY(G; H)$.

$l(H; G) := (r_*(H; G))/[N_G(H): H]$ where $N_G(H)$ means the normalizer of H in G.

We say that G satisfies the *RH-condition* if G satisfies the E condition and $l(H; G)$ is a non-negative integer for any $H \in CY(G)$.

We denote by D_8 and Q_8, respectively, the dihedral group of order 8 and quaternion group,

$$\text{i.e.,} \quad D_8 = \langle a, b; a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle,$$
$$Q_8 = \langle a, b; a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle.$$

The character table of D_8 is as follows (Q_8 has the same character table):

	1	a	a^2	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	1	−1	−1
χ_3	1	−1	1	1	−1
χ_4	1	−1	1	−1	1
χ_5	2	0	−2	0	0

§2. **Results.** We have the following results.

Proposition 1. *Let G be a finite subgroup of $GL(g, C)$ which is isomorphic to D_8, and let χ_G be the character of the natural representation $G \to GL(g, C)$. Let $n_1\chi_1 + \cdots + n_5\chi_5$ be the decomposition into irreducible characters of χ_G. Then G satisfies the CY-and RH-conditions if and only if n_i's satisfy the following relations*:

(1) $1 \geq n_1 + n_2 - n_3 - n_4$ (2) $n_5 \geq n_3 + n_4$

(3) $1 \geq n_1 - n_2 + n_3 - n_4$ (4) $1 \geq n_1 - n_2 - n_3 + n_4$.

Proposition 2. *Let G be a finite subgroup of $GL(g, C)$ which is isomorphic to Q_8, and let χ_G be the character of the natural representation $G \to GL(g, C)$. Let $n_1\chi_1 + \cdots + n_5\chi_5$ be the decomposition into irreducible characters of χ_G. Then G satisfies the CY-and RH-conditions if and only if n_i's satisfy the following relations*:

(1) $1 \geq n_1 + n_2 - n_3 - n_4$ (2) $n_1 - n_2 - n_3 - n_4 + n_5 \geq 1$

(3) $1 \geq n_1 - n_2 + n_3 - n_4$ (4) $1 \geq n_1 - n_2 - n_3 + n_4$.

Remark. In Propositions 1, 2, we have $g = n_1 + n_2 + n_3 + n_4 + 2n_5$. By using of these propositions, we obtain the following.

Theorem 1. *Assume that $G(\simeq D_8) \subset GL(g, C)$ satisfies the CY-and RH-conditions. If G does not arise from a compact Riemann surface of genus g, then $g \equiv 1$ (mod 4) and $n_1 = 1$, $n_2 = n_3 = n_4 = 0$, $n_5 \equiv 0$ (mod 2).*

Theorem 2. *Assume that $G(\simeq Q_8) \subset GL(g, C)$ satisfies the CY-and RH-conditions. If G does not arise from a compact Riemann surface of genus g, then $g \equiv 1$ (mod 4) and $n_1 = 1$, $n_2 = n_3 = n_4 = 0$, $n_5 \equiv 0$ (mod 2).*

References

[1] A. Kuribayashi and H. Kimura: Automorphism groups of compact Riemann surfaces of genus five. J. Algebra., **134**, 80–103 (1990).

[2] I. Kuribayashi: On an algebraization of the Riemann-Hurwitz relation. Kodai. Math., **7**, 222–237 (1984).