88. (2, 15)-torus Knot is not Slice in CP^2

By Akira YASUHARA
Department of Mathematics, School of Science and Engineering, Waseda University

(Communicated by Shokichi IYANAGA, M. J. A., Dec. 12, 1991)

§ 1. **Introduction.** Unless otherwise stated all manifolds and maps are smooth.

Let M be a closed 4-manifold and K a knot in $\partial(M-\mathrm{Int}\,B^4)\cong S^3$ where B^4 is an embedded 4-ball in M. If K bounds a properly embedded 2-disk in $M-\mathrm{Int}\,B^4$, then we call the knot K a *slice knot* in M. Let $Slice(M)$ be the set of slice knots in M. We note that $Slice(S^4)$ is unequal to the set of knots in S^3 (Fox and Milnor [1]) and $Slice(S^4)$ is a subset of $Slice(M)$. In [5], Suzuki proved that $Slice(S^2\times S^2)$ is equal to the set of knots in S^3, and asked the question "Is there a 4-manifold M such that $Slice(M)$ is equal to neither $Slice(S^4)$ nor the set of knots in S^3?". In this paper we shall prove the following theorem.

Theorem. *The set $Slice(CP^2)$ does not contain a* (2, 15)*-torus knot.*

It is easily seen that $Slice(S^4)$ is a proper subset of $Slice(CP^2)$ (for example, see Kervaire and Milnor [2]). Thus this theorem gives an affirmative answer to Suzuki's question.

The author wishes to thank Professor Shin'ichi Suzuki for his encouragement.

§ 2. **Preliminaries.** In this section M means an oriented, connected, simply connected, closed 4-manifold. We need the following four lemmas to prove Theorem.

Lemma 1 (Rohlin [4]). *If $\xi\in H_2(M;Z)$ is represented by an embedded 2-sphere in M, then*

(a) $\left|\frac{\xi^2}{2}-\sigma(M)\right|\leq \mathrm{rank}\,H_2(M;Z)$ *if ξ is divisible by* 2,

(b) $\left|\frac{\xi^2(p^2-1)}{2p^2}-\sigma(M)\right|\leq \mathrm{rank}\,H_2(M;Z)$ *if ξ is divisible by an odd prime p,*

where ξ^2 is the self-intersection number of ξ and $\sigma(M)$ is the signature of M.

Lemma 2 (Kervaire and Milnor [2]). *Let $\xi\in H_2(M;Z)$ be dual to the Stiefel-Whitney class $w_2(M)$. If ξ is represented by an embedded* 2*-sphere in M, then $\xi^2\equiv\sigma(M)$ mod 16.*

Lemma 3 (Weintraub [6], Yamamoto [7]). *Suppose $\xi\in H_2(M-\mathrm{Int}\,B^4, \partial(M-\mathrm{Int}\,B^4);Z)$ is represented by a properly embedded* 2*-disk Δ in $M-\mathrm{Int}\,B^4$ and let K be a knot $\partial\Delta\subset\partial(M-\mathrm{Int}\,B^4)$. If the unknotting number of*

K is u, then ξ is represented by an embedded 2-sphere in $M\sharp u(CP^2\sharp\overline{CP^2})$. Here ξ is identified with its image

$$H_2(M-\operatorname{Int}B^4, \partial(M-\operatorname{Int}B^4); Z) \xleftarrow{\cong} H_2(M-\operatorname{Int}B^4; Z) \longrightarrow H_2(M\sharp u(CP^2\sharp\overline{CP^2}); Z).$$

Lemma 4 (Kuga [3]). *Suppose M has the intersection form*

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \oplus \langle 1 \rangle,$$

with respect to generators α, β, γ of $H_2(M; Z)\cong Z\oplus Z\oplus Z$. If $x\geq 2$, $y\geq 2$, and $z^2=1$, then the homology class $x\alpha+y\beta+z\gamma$ cannot be represented by an embedded 2-sphere in M.

§ 3. Proof of Theorem. Let

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \oplus \langle 1 \rangle$$

be the intersection form of $S^2\times S^2\sharp CP^2$ with respect to generators α, β, γ of $H_2(S^2\times S^2\sharp CP^2; Z)\cong Z\oplus Z\oplus Z$. There exist mutually disjoint ten properly embedded 2-disks $\Delta_1, \cdots, \Delta_{10}$ such that $\Delta_1\cup\cdots\cup\Delta_{10}$ represents $2\alpha+8\beta\in H_2(S^2\times S^2-\operatorname{Int}B_1^4, \partial(S^2\times S^2-\operatorname{Int}B_1^4); Z)$ and such that $\partial(\Delta_1\cup\cdots\cup\Delta_{10})\subset\partial(S^2\times S^2-\operatorname{Int}B^4)$ is the link as illustrated by Fig. 1. It is not hard to see that nine strips $b_1, \cdots, b_9$ connecting the 2-disks $\Delta_1, \cdots, \Delta_{10}$ can be chosen so that $D_1=\Delta_1\cup\cdots\cup\Delta_{10}\cup b_1\cup\cdots\cup b_9$ is an embedded 2-disk in $S^2\times S^2-\operatorname{Int}B_1^4$ and so that $\partial D_1\subset\partial(S^2\times S^2-\operatorname{Int}B_1^4)$ is a $(-2, 15)$-torus knot as illustrated by Fig. 2. Thus this $(-2, 15)$-torus knot bounds the embedded 2-disk D_1 which represents $2\alpha+8\beta\in H_2(S^2\times S^2-\operatorname{Int}B^4, \partial(S^2\times S^2-\operatorname{Int}B^4); Z)$.

Suppose *Slice*(CP^2) contains a $(2, 15)$-torus knot, then a $(2, 15)$-torus

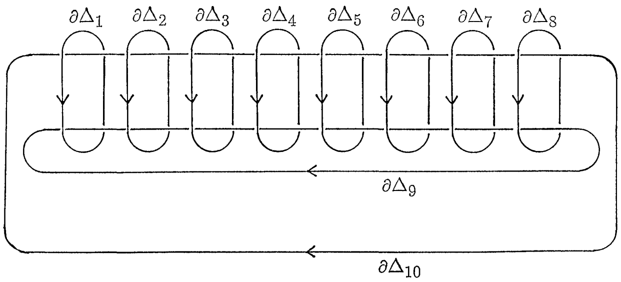

Fig. 1

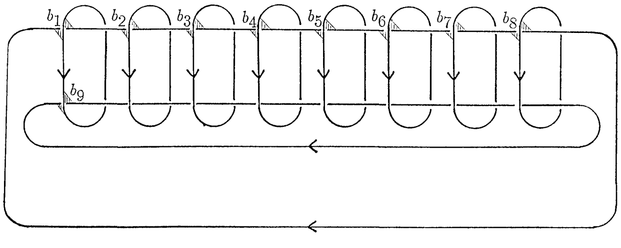

Fig. 2

knot bounds a properly embedded 2-disk D_2 in $CP^2-\mathrm{Int}\,B_2^4$. This implies that there exists an integer z such that $z\gamma \in H_2(CP^2-\mathrm{Int}\,B_2^4, \partial(CP^2-\mathrm{Int}\,B_2^4); Z)$ is represented by the properly embedded 2-disk D_2 in $CP^2-\mathrm{Int}\,B_2^4$. Since there exists an orientation reversing diffeomorphism from the pair $(\partial(S^2\times S^2-\mathrm{Int}\,B_1^4), \partial D_1)$ to the pair $(\partial(CP^2-\mathrm{Int}\,B_2^4), \partial D_2)$, $2\alpha+8\beta+z\gamma \in H_2(S^2\times S^2\# CP^2)$ can be represented by the embedded 2-sphere $D_1\cup D_2$ in $S^2\times S^2\# CP^2$.

If z is even, then $2\alpha+8\beta+z\gamma$ is divisible by 2. By Lemma 1, we have

$$\left|\frac{-32+z^2}{2}-1\right|\leq 3. \tag{1}$$

Moreover, by using the fact that the unknotting number of a (2, 15)-torus knot is 7 and Lemma 3, we find that $z\gamma$ is represented by an embedded 2-sphere in $CP^2\# 7(CP^2\#\overline{CP^2})$. By Lemma 1, we have

$$\left|\frac{z^2}{2}-1\right|\leq 15. \tag{2}$$

We note that there is no even integer z which satisfies inequalities (1) and (2). Therefore z is not even. That is, either $z^2=1$ or z is divisible by an odd prime p. In the latter case, since $z\gamma$ is represented by an embedded 2-sphere in $CP^2\# 7(CP^2\#\overline{CP^2})$,

$$\left|\frac{z^2(p^2-1)}{2p^2}-1\right|\leq 15,$$

by Lemma 1. It follows that

$$z^2\leq 32\left(1+\frac{1}{p^2-1}\right)\leq 36.$$

This implies $z^2=9$ or 25.

On the other hand, since $2\alpha+8\beta+z\gamma$ is dual to $w_2(S^2\times S^2\# CP^2)$, $z^2\equiv 1 \bmod 16$ by Lemma 3. Therefore we have $z^2=1$. However, $z^2\neq 1$ by Lemma 4, a contradiction. Hence *Slice*(CP^2) does not contain a (2, 15)-torus knot. This completes the proof.

References

[1] R. H. Fox and J. W. Milnor: Singularities of 2-spheres in 4-space and cobordism of knots. Osaka J. Math., **3**, 257–267 (1966).

[2] M. A. Kervaire and J. W. Milnor: On 2-spheres in 4-manifolds. Proc. Nat. Acad. Sci. U.S.A., **47**, 1651–1657 (1961).

[3] K. Kuga: Representing homology classes of $S^2\times S^2$. Topology, **23**, 133–137 (1984).

[4] V. A. Rohlin: Two-dimensional submanifolds of four-dimensional manifolds. Functional Anal. Appl., **5**, 39–48 (1974).

[5] S. Suzuki: Local knots of 2-spheres in 4-manifolds. Proc. Japan Acad., **45**, 34–38 (1969).

[6] S. H. Weintraub: Inefficiently embedded surfaces in 4-manifolds. Algebraic Topology Aarhus 1978 (eds. J. L. Dupont and I. H. Madsen). Lect. Notes in Math., vol. 763, Springer-Verlag, Berlin, New York (1979).

[7] M. Yamamoto: Lower bounds for the unknotting numbers of certain torus knots. Proc. Amer. Math. Soc., **86**, 519–524 (1982).