9. *One Criterion for Multivalent Functions*

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Let P be the class of functions $p(z)$ which are analytic in the unit disk $E=\{z: |z|<1\}$, with $p(0)=1$ and $\mathrm{Re}\, p(z)>0$ in E.

If $p(z) \in P$, we say $p(z)$ is a *Carathéodory function*. It is well-known that if $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ is analytic in E and $f'(z) \in P$, then $f(z)$ is univalent in E [1, 8].

Ozaki [5, Theorem 2] extended the above result to the following:

If $f(z)$ is analytic in a convex domain D and

$$\mathrm{Re}\,(e^{i\alpha} f^{(p)}(z))>0 \qquad \text{in } D$$

where α is a real constant, then $f(z)$ is at most p-valent in D.

This shows that if $f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n$ is analytic in E and

$$\mathrm{Re}\, f^{(p)}(z)>0 \qquad \text{in } E,$$

then $f(z)$ is p-valent in E.

Nunokawa improved the above result to the following:

Theorem A. *Let $p \geqq 2$. If $f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n$ is analytic in E and*

$$|\arg f^{(p)}(z)|<\frac{3}{4}\pi \qquad \text{in } E,$$

then $f(z)$ is p-valent in E (cf. [3]).

Theorem B. *Let $p \geqq 2$. If $f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n$ is analytic in E and*

$$\mathrm{Re}\, f^{(p)}(z)>-\frac{\log(4/e)}{2\log(e/2)}\, p! \qquad \text{in } E,$$

then $f(z)$ is p-valent in E (cf. [4]).

In this paper, we need the following lemmas.

Lemma 1 ([6], Lemma 4). *Let $p(z)$ be analytic in E with $p(0)=1$ and $\mathrm{Re}\, p(z)>1/2$ in E.*

Then for any function $f(z)$, analytic in E, the function $p(z)*f(z)$ takes its values in the convex hull of $f(z)$, where $p(z)*f(z)$ denotes the convolution or Hadamard product of $p(z)$ with $f(z)$.

Lemma 2 ([7]). *Let $p(z)$ be analytic in E with $p(0)=1$. Suppose that $\alpha>0$, $\beta<1$ and that for $z \in E$, $\mathrm{Re}\,(p(z)+\alpha z p'(z))>\beta$.*

Then for $z \in E$,

$$\mathrm{Re}\, p(z)>1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\alpha n}.$$

The estimate is best possible for

$$p_0(z)=2\beta-1+2(1-\beta)\sum_{n=1}^{\infty}\frac{z^n(-1)^n}{1+\alpha n}.$$

Proof. For $z \in E$, write $p(z)=1+\sum_{n=1}^{\infty} p_n z^n$, so that

$$\mathrm{Re}\left\{1+\sum_{n=1}^{\infty}(1+\alpha n)p_n z^n\right\}>\beta.$$

Thus

$$\mathrm{Re}\left\{1+\frac{1}{2(1-\beta)}\sum_{n=1}^{\infty}(1+\alpha n)p_n z^n\right\}>\frac{1}{2}.$$

Now

$$p(z)=\left\{1+\frac{1}{2(1-\beta)}\sum_{n=1}^{\infty}(1+\alpha n)p_n z^n\right\}*\left\{1+2(1-\beta)\sum_{n=1}^{\infty}\frac{z^n}{1+\alpha n}\right\}$$

and so by Lemma 1,

$$\mathrm{Re}\,p(z)>1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\alpha n}$$

as required. Simple substitution for $p_0(z)$ shows that the result is best possible.

Remark. In Lemma 2, if we put

$$A(\alpha)=\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\alpha n}, \qquad \alpha>0,$$

then we easily have

$$A(1)=\log(2/e) \quad \text{and} \quad A(1/2)=\log(e/4).$$

Lemma 3 ([2], Theorem 8). *Let $f(z)=z^p+\sum_{n=p+1}^{\infty}a_n z^n$ be analytic in E and if there exists a $(p-k+1)$-valent starlike function $g(z)=z^{p-k+1}+\sum_{n=p-k+2}^{\infty}b_n z^n$ that satisfies*

$$\mathrm{Re}\,\frac{zf^{(k)}(z)}{g(z)}>0 \qquad \text{in } E,$$

then $f(z)$ is p-valent in E.

Main theorem. *Let $p\geqq 3$. If $f(z)=z^p+\sum_{n=p+1}^{\infty}a_n z^n$ is analytic in E and*

(1) $$\mathrm{Re}\,f^{(p)}(z)>-\frac{1-4(\log(4/e))(\log(e/2))}{4(\log(4/e))(\log(e/2))}p! \qquad \text{in } E,$$

then $f(z)$ is p-valent in E.

Proof. Let us put

$$p(z)=f^{(p-1)}(z)/(p!\,z).$$

Then, from the assumption (1) and by an easy calculation, we have

(2) $$\mathrm{Re}\,(p(z)+zp'(z))=\mathrm{Re}\,(f^{(p)}(z)/p!)$$
$$>-\frac{1-4(\log(4/e))(\log(e/2))}{4(\log(4/e))(\log(e/2))} \qquad \text{in } E,$$

and $p(0)=1$.

Then, from (2) and Lemma 2, we have

(3) $$\mathrm{Re}\,p(z)=\frac{1}{p!}\,\mathrm{Re}\,\frac{f^{(p-1)}(z)}{z}$$
$$>\frac{\log(e^3/16)}{2\log(e/4)}\doteqdot-0.2943496\cdots \qquad \text{in } E.$$

Next, let us put

$$q(z)=2f^{(p-2)}(z)/(p!\,z^2).$$

Then, from (3) and by an easy calculation, we have

$$(4)\qquad \mathrm{Re}\left(q(z)+\frac{1}{2}zq'(z)\right)=\frac{1}{p\,!}\,\mathrm{Re}\,\frac{f^{(p-1)}(z)}{z}$$
$$>\frac{\log(e^3/16)}{2\log(e/4)}\qquad \text{in } E,$$

and $q(0)=1$.

Then, from (4) and Lemma 2, we have

$$\mathrm{Re}\,q(z)=\frac{2}{p\,!}\,\mathrm{Re}\,\frac{f^{(p-2)}(z)}{z^2}>0\qquad \text{in } E.$$

This shows that

$$\mathrm{Re}\,\frac{zf^{(p-2)}(z)}{z^3}>0\qquad \text{in } E.$$

It is trivial that $g(z)=z^3$ is 3-valently starlike in E. Therefore, from Lemma 3, we see that $f(z)$ is p-valent in E. This completes our proof.

Remark. We have

$$\frac{\log(4/e)}{2\log(e/2)}\doteqdot 0.62944\cdots$$

and

$$\frac{1-4(\log(4/e))(\log(e/2))}{4(\log(4/e))(\log(e/2))}\doteqdot 1.10907\cdots.$$

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