83. *Embedding into Kac-Moody Algebras and Construction of Folding Subalgebras for Generalized Kac-Moody Algebras*

By Satoshi NAITO
Department of Mathematics, Kyoto University

(Communicated by Shokichi IYANAGA, M. J. A., Dec. 12, 1991)

Introduction. In the preceding paper [5], we defined a *regular subalgebra* $\bar{\mathfrak{g}}$ of a symmetrizable *Kac-Moody algebra* $\mathfrak{g}(A)$, and showed that $\bar{\mathfrak{g}}$ is isomorphic to a *generalized Kac-Moody algebra* (=GKM algebra) $\mathfrak{g}(\bar{A})$ associated to a canonically defined symmetrizable GGCM $\bar{A}$, as explained below.

In the first half of this paper, we show that a symmetrizable GKM algebra $\mathfrak{g}(A)$ can be embedded into some Kac-Moody algebra as a regular subalgebra under a certain weak condition on the GGCM A. In the latter half of this paper, we introduce and study what we call a *folding subalgebra* of a symmetrizable GKM algebra $\mathfrak{g}(A)$, corresponding to a diagram automorphism π of the GGCM A. This subalgebra is contained in the fixed point subalgebra of an automorphism of $\mathfrak{g}(A)$ induced by π, and is easier to deal with than the fixed point subalgebra itself.

§ 1. Embedding of GKM algebras into Kac-Moody algebras.

1.1. Regular subalgebras. Here, we recall the notion of regular subalgebras of symmetrizable Kac-Moody algebras introduced in [5]. For the detailed accounts, see [2], [5], and [6]. Let $\mathfrak{g}(A)$ be a Kac-Moody algebra associated to a symmetrizable *generalized Cartan matrix* (=GCM) A over the complex number field $\boldsymbol{C}$, and $\mathfrak{h}$ its *Cartan subalgebra.*

Definition 1.1 ([5]). A subset $\bar{\Pi}=\{\beta_r\}_{r=1}^m$ of the root system Δ of $\mathfrak{g}(A)$ is called *fundamental* if it satisfies the following:

(1) $\beta_1, \beta_2, \cdots, \beta_m$ are linearly independent;

(2) $\beta_i-\beta_j \notin \Delta \quad (1\leq i\neq j\leq m)$;

(3) if β_i is an *imaginary root*, then it is a positive root.

For each imaginary root β_i, we define $\beta_i^\vee:=\nu^{-1}(\beta_i)$, where $\nu: \mathfrak{h}\to\mathfrak{h}^*$ is a linear isomorphism determined by a *standard invariant form* $(\cdot|\cdot)$ on $\mathfrak{g}(A)$. For real root β_i, $\beta_i^\vee$ has been defined as a dual real root of β_i. Then, we proved in [5] that $\bar{A}:=(\bar{a}_{ij})_{i,j=1}^m$ with $\bar{a}_{ij}=\langle\beta_j, \beta_i^\vee\rangle$ is a symmetrizable *generalized GCM* (=GGCM), that is, $\bar{A}$ satisfies the following:

(C1) either $\bar{a}_{ii}=2$ or $\bar{a}_{ii}\leq 0$;

(C2) $\bar{a}_{ij}\leq 0$ if $i\neq j$, and $\bar{a}_{ij}\in \boldsymbol{Z}$ if $\bar{a}_{ii}=2$;

(C3) $\bar{a}_{ij}=0$ implies $\bar{a}_{ji}=0$.

Now, take and fix non-zero root vectors $E_r\in\mathfrak{g}_{\beta_r}$ and $F_r\in\mathfrak{g}_{-\beta_r}$ such that $[E_r, F_r]=\beta_r^\vee$ $(1\leq r\leq m)$. Then,

Theorem 1.1 ([5]). *Let $\bar{\mathfrak{g}}$ be a subalgebra of $\mathfrak{g}(A)$ generated by E_r, F_r $(1 \leq r \leq m)$, and a vector subspace $\mathfrak{h}_0$ of $\mathfrak{h}$ such that the triple $(\mathfrak{h}_0, \{\beta_r | \mathfrak{h}_0\}_{r=1}^m, \{\beta_r^\vee\}_{r=1}^m)$ is a realization of the above* GGCM $\bar{A}$. *Then, $\bar{\mathfrak{g}}$ is canonically isomorphic to a* GKM *algebra $\mathfrak{g}(\bar{A})$.*

This subalgebra $\bar{\mathfrak{g}}$ is called a *regular subalgebra* of $\mathfrak{g}(A)$.

1.2. Existence of an embedding. The symmetrizable GGCM $\bar{A}$ associated to the GKM algebra $\mathfrak{g}(\bar{A})$, isomorphic to the regular subalgebra $\bar{\mathfrak{g}}$, satisfies the following *integrality condition* (INT):

(INT) *For each i with $\bar{a}_{ii} \leq 0$, there exists a positive real number $\bar{\eta}_i$ such that $\bar{\eta}_i \cdot \bar{a}_{ij} \in \boldsymbol{Z}$ for every j $(1 \leq j \leq m)$.*

Here, we consider a converse problem.

Problem. *Let $\bar{A} = (\bar{a}_{ij})_{i,j=1}^m$ be an arbitrary symmetrizable* GGCM *with the above integrality* (INT). *Then, can we embed the* GKM *algebra $\mathfrak{g}(\bar{A})$ into some Kac-Moody algebra as a regular subalgebra?*

Note that, in considering the above problem, we can and do assume that the GGCM $\bar{A} = (\bar{a}_{ij})_{i,j=1}^m$ satisfies the following for some p and q such that $p+q=m$:

(G1) $\bar{a}_{ii} = 2$ $(1 \leq i \leq p)$,

(G2) $\bar{a}_{jj} \leq 0$ $(p+1 \leq j \leq p+q=m)$,

(G3) $\bar{a}_{ij} \in \boldsymbol{Z}$ $(1 \leq i, j \leq m)$.

Then, we have the following theorem, which answers the above problem affirmatively.

Theorem 1.2. *Let $\bar{A} = (\bar{a}_{ij})_{i,j=1}^m$ be a symmetrizable* GGCM *with the integrality* (INT). *And assume that $\bar{A}$ satisfies* (G1)–(G3). *Then, the* GKM *algebra $\mathfrak{g}(\bar{A})$ is isomorphic to a regular subalgebra $\bar{\mathfrak{g}}$ of the Kac-Moody algebra $\mathfrak{g}(A)$ associated to the $2m \times 2m$ symmetrizable* GCM *given below*:

$A = (a_{ij})_{i,j=1}^{2m}$, *with*

$$\begin{bmatrix} a_{2k-1,2l-1} & a_{2k-1,2l} \\ a_{2k,2l-1} & a_{2k,2l} \end{bmatrix} := \begin{bmatrix} \bar{a}_{kl} & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{pmatrix} 1 \leq k \leq p \\ 1 \leq l \leq m, \quad l \neq k \end{pmatrix},$$

$$\begin{bmatrix} a_{2k-1,2k-1} & a_{2k-1,2k} \\ a_{2k,2k-1} & a_{2k,2k} \end{bmatrix} := \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad (1 \leq k \leq p),$$

$$\begin{bmatrix} a_{2k-1,2l-1} & a_{2k-1,2l} \\ a_{2k,2l-1} & a_{2k,2l} \end{bmatrix} := \begin{bmatrix} 2\bar{a}_{kl} \cdot u_k & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{pmatrix} p+1 \leq k \leq p+q=m \\ 1 \leq l \leq m, \quad l \neq k \end{pmatrix},$$

$$\begin{bmatrix} a_{2k-1,2k-1} & a_{2k-1,2k} \\ a_{2k,2k-1} & a_{2k,2k} \end{bmatrix} := \begin{bmatrix} 2 & v_k \\ v_k & 2 \end{bmatrix} \quad (p+1 \leq k \leq p+q=m),$$

where $u_k := \begin{cases} -\bar{a}_{kk} & (\bar{a}_{kk} \neq 0) \\ 1 & (\bar{a}_{kk} = 0) \end{cases}, \qquad v_k := \bar{a}_{kk} - 2 \quad (p+1 \leq k \leq p+q=m).$

Sketch of proof. Put $\beta_r := \alpha_{2r-1} + \alpha_{2r}$ $(1 \leq r \leq m = p+q)$, where $\{\alpha_r\}_{r=1}^{2m}$ is the simple root system of the Kac-Moody algebra $\mathfrak{g}(A)$. Then, $\{\beta_r\}_{r=1}^m$ is a fundamental subset of the root system Δ of $\mathfrak{g}(A)$. Therefore, we see from Theorem 1.1 that there exists a regular subalgebra $\bar{\mathfrak{g}}$ of $\mathfrak{g}(A)$, which is canonically isomorphic to a GKM algebra $\mathfrak{g}(\tilde{A})$ associated to the GGCM $\tilde{A} := (\langle \beta_j, \beta_i^\vee \rangle)_{i,j=1}^m$. On the other hand, we have

$$\langle \beta_j, \beta_i^\vee \rangle = \begin{cases} 2(\beta_i \mid \beta_j)/(\beta_i \mid \beta_i) & \text{if } \beta_i \text{ is a real root} \\ (\beta_i \mid \beta_j) & \text{if } \beta_i \text{ is an imaginary root,} \end{cases}$$

where $(\cdot \mid \cdot)$ is a standard invariant form on $\mathfrak{g}(A)$. So, we can show that, for a suitably chosen standard invariant form $(\cdot \mid \cdot)$ on $\mathfrak{g}(A)$, $\tilde{A} = \check{D}\overline{A}$, where $\check{D}$ is an invertible diagonal matrix. Therefore, $\mathfrak{g}(\overline{A})$ is isomorphic to $\mathfrak{g}(\tilde{A})$ by rescaling the Chevalley generators. Hence we get the theorem.

§ 2. Folding subalgebras of a GKM algebra.

2.1. Diagram automorphisms of a GGCM. Let $A = (a_{ij})_{i,j=1}^n$ be a GGCM.

Definition 2.1. A permutation π on $I := \{1, 2, \cdots, n\}$ is called a *diagram automorphism* of a GGCM $A = (a_{ij})_{i,j=1}^n$ if

$$a_{\pi(i),\pi(j)} = a_{ij} \qquad \text{for every } i, j \ (1 \leq i, j \leq n).$$

Since a diagram automorphism π is a permutation on I, we have a unique decomposition of I into its disjoint subsets I_j $(1 \leq j \leq m)$, such that the restriction of π to I_j is a cyclic permutation $(1 \leq j \leq m)$.

Lemma 2.1. *For every* j_1, j_2 $(1 \leq j_1, j_2 \leq m)$ *and* $i_1, i_2 \in I_{j_1}$, *we have* $\sum_{k \in I_{j_2}} a_{k,i_1} = \sum_{k \in I_{j_2}} a_{k,i_2}$.

In view of Lemma 2.1, we set $\bar{a}_{ij} := \sum_{k \in I_i} a_{kl}$ for $l \in I_j$ $(1 \leq i, j \leq m)$, which does not depend on the choice of $l \in I_j$.

Lemma 2.2. *If* $\bar{a}_{ii} = \sum_{k \in I_i} a_{kl}$, $l \in I_i$, *is a positive real number, then the Dynkin diagram* $S(A^i)$ *of the principal submatrix* $A^i := (a_{kl})_{k,l \in I_i}$ *of* A *is either of the following two forms*:

Case (I). *$S(A^i)$ is a disjoint union of Dynkin diagrams of type* A_1;

Case (II). *$S(A^i)$ is a disjoint union of Dynkin diagrams of type* A_2.

We now consider the case where $A = (a_{ij})_{i,j=1}^n$ is an indecomposable, symmetrizable GGCM. Fix a decomposition of A: $A = DB$, where $D = \operatorname{diag}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ $(\varepsilon_i > 0, 1 \leq i \leq n)$, and ${}^tB = B$.

Lemma 2.3. *Let* π *be a diagram automorphism of an indecomposable, symmetrizable* GGCM A. *Then, we have* $\varepsilon_{\pi(i)} = \varepsilon_i$ *for every* i $(1 \leq i \leq n)$.

2.2. Construction of folding subalgebras. Let $A = (a_{ij})_{i,j=1}^n$ be an indecomposable, symmetrizable GGCM.

Now, for i $(1 \leq i \leq m)$, we say "Case $X(i)$" in the case where $\bar{a}_{ii} \leq 0$, or in the case where $\bar{a}_{ii} > 0$ and Case (I) in Lemma 2.2 happens. And we say "Case $Y(i)$" in the case where $\bar{a}_{ii} > 0$ and Case (II) in Lemma 2.2 happens. Then, we put

$$\hat{A} := (\hat{a}_{ij})_{i,j=1}^m \qquad \text{with } \hat{a}_{ij} := \begin{cases} \bar{a}_{ij} & \text{in Case } X(i) \\ 2\bar{a}_{ij} & \text{in Case } Y(i). \end{cases} \tag{2.1}$$

Moreover, we put for i $(1 \leq i \leq m)$,

$$H_i := \textstyle\sum_{k \in I_i} \alpha_k^\vee, \quad E_i := \sum_{k \in I_i} e_k, \quad F_i := \sum_{k \in I_i} f_k \qquad \text{in Cass } X(i),$$

$$H_i := 2(\textstyle\sum_{k \in I_i} \alpha_k^\vee), \quad E_i := \sqrt{2}(\sum_{k \in I_i} e_k), \quad F_i := \sqrt{2}(\sum_{k \in I_i} f_k) \qquad \text{in Case } Y(i),$$

where e_i, f_i $(i \in I)$ are the Chevalley generators, and $\{\alpha_i^\vee\}_{i \in I}$ is the set of all simple coroots of the GKM algebra $\mathfrak{g}(A)$.

Proposition 2.1. *$\hat{A}$ is an indecomposable, symmetrizable* GGCM.

Remark 2.1. Even if A is a GCM, $\hat{A}$ is not a GCM except for the case where, for every i $(1 \leq i \leq m)$, Case (I) or Case (II) in Lemma 2.2 happens.

Let $\hat{\mathfrak{g}}$ be a subalgebra of $\mathfrak{g}(A)$ generated by E_i, F_i, and H_i $(1 \leq i \leq m)$. Note that $\hat{\mathfrak{g}}$ is actually contained in the derived subalgebra $[\mathfrak{g}(A), \mathfrak{g}(A)]$ of the GKM algebra $\mathfrak{g}(A)$, since E_i, F_i, and H_i $(1 \leq i \leq m)$ lie in it.

Definition 2.2. We call the above subalgebra $\hat{\mathfrak{g}}$ the *folding subalgebra* of $\mathfrak{g}(A)$ corresponding to a diagram automorphism π of A.

Theorem 2.1. *Any folding subalgebra of* $\mathfrak{g}(A)$ *is canonically isomorphic to the derived algebra of a symmetrizable* GKM *algebra. Let* $\hat{\mathfrak{g}}$ *be a folding subalgebra of* $\mathfrak{g}(A)$ *generated by* E_i, F_i, *and* H_i $(1 \leq i \leq m)$. *Then, the canonical isomorphism* Φ *of the derived algebra* $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$ *onto* $\hat{\mathfrak{g}}$ *is given as:*

$$\Phi(\hat{e}_i) = E_i, \quad \Phi(\hat{f}_i) = F_i, \quad \text{and} \quad \Phi(\hat{\alpha}_i^\vee) = H_i \quad (1 \leq i \leq m).$$

Here $\hat{e}_i$, $\hat{f}_i$ $(1 \leq i \leq m)$ *are the Chevalley generators, and* $\{\hat{\alpha}_i^\vee\}_{i=1}^m$ *is the set of all simple coroots of the* GKM *algebra* $\mathfrak{g}(\hat{A})$ *associated to the* GGCM $\hat{A}$ *in* (2.1).

Remark 2.2. From the above theorem, we see that, under the operation of making folding subalgebras, the category of the derived algebras of Kac-Moody algebras is not closed (see Remark 2.1), but that for GKM algebras is closed.

2.3. Type of the GGCM $\hat{A}$. First, note that the *classification theorem* in [2, Chap. 4] for GCMs also holds in the case of indecomposable GGCMs, except that there exists one additional affine matrix—the zero 1×1 matrix. We call an indecomposable GGCM A *of hyperbolic type*, if it is symmetrizable, of indefinite type, and if every proper indecomposable principal submatrix of A is a GGCM of finite or affine type. Then, we have the following theorem.

Theorem 2.2. *Let* $A = (a_{ij})_{i,j=1}^n$ *be an indecomposable, symmetrizable* GGCM, *and* $\hat{A}$ *a* GGCM *defined in* (2.1). *If* A *is a* GGCM *of finite* (*resp. affine or indefinite*) *type, then* $\hat{A}$ *is again a* GGCM *of finite* (*resp. affine or indefinite*) *type. Further, if* A *is a* GGCM *of hyperbolic type, then* $\hat{A}$ *is again a* GGCM *of hyperbolic type.*

Remark 2.3. In the case where A is a GCM of affine type, we can actually determine all $\hat{A}$ by the list of diagram automorphisms of A in [1]. And in the case of hyperbolic type GCM, all $\hat{A}$ can be again determined, using the list of all hyperbolic type GCMs in [3] (see [4] for details).

2.4. The complete reducibility. For integrable highest weight modules of the derived algebra of a Kac-Moody algebra, we have the following complete reducibility with respect to its folding subalgebras.

Theorem 2.3. *Let* $A = (a_{ij})_{i,j=1}^n$ *be an indecomposable, symmetrizable* GCM, $\Lambda \in (\sum_{i=1}^n \mathbf{C}\alpha_i^\vee)^*$ *a dominant integral weight, and* $L(\Lambda)$ *an integrable highest weight module with highest weight* Λ *over the derived Kac-Moody algebra* $[\mathfrak{g}(A), \mathfrak{g}(A)]$. *Assume that* $\hat{A}$ *is again a* GCM. *Then, as* $\hat{\mathfrak{g}}$*-modules,*

$L(\Lambda)$ is isomorphic to a direct sum of $[\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$*-modules* $L(\lambda)$ *such that* $\lambda \in (\sum_{i=1}^{m} \boldsymbol{C}\hat{\alpha}_i^\vee)^*$, $\langle \lambda, \hat{\alpha}_i^\vee \rangle \in \boldsymbol{Z}_{\geq 0}$ $(1 \leq i \leq m)$, *under the identification* $\hat{\mathfrak{g}} \cong [\mathfrak{g}(\hat{A}), \mathfrak{g}(\hat{A})]$.

Acknowledgements. I would like to thank Professor Takeshi Hirai for his valuable suggestions. I would also like to thank Professor Jun Morita for his encouragement.

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