81. *Remarks on Viscosity Solutions for Evolution Equations*

By Yun-Gang CHEN,*) Yoshikazu GIGA,**) and Shun'ichi GOTO***)

(Communicated by Kunihiko KODAIRA, M. J. A., Dec. 12, 1991)

1. Introduction. We consider a degenerate parabolic equation

$$\partial u/\partial t+F(t, x, u, \nabla u, \nabla^2 u)=0, \tag{1}$$

where ∇ stands for the spatial derivatives. We are concerned with a viscosity subsolution which needs not to be continuous. We say a function $u(t,x)$ defined in a parabolic neighborhood of (t_0, x_0) is *left accessible* at (t_0, x_0) if there are sequences $x_l \to x_0$, $t_l \to t_0$ with $t_l < t_0$ such that $\lim_{l\to\infty} u(t_l, x_l) = u(t_0, x_0)$. Our goal is to show that a viscosity subsolution is left accessible at each (parabolic) interior point of the domain of definition for a wide class of F. We also clarify the relation between viscocity subsolutions defined on time interval $(0, T)$ and those on $(0, T]$. Similar problems are studied in other contexts by Crandall and Newcomb [3] and by Ishii [7]. We thank Professor Hitoshi Ishii for pointing out these references.

There are technical errors in the proof of Ishii's lemma up to the terminal time in our previous work [1, Lemma 3.1 and Proposition 3.2]. If we note left accessibility, the proof can be easily fixed. We take this opportunity to correct technical errors in [1] somewhat related to left accessibility. We thank Professor Joseph Fu for pointing out a couple of errors in the proof of [1, Lemma 3.1 and Proposition 3.2].

For $h: L \to \boldsymbol{R}$ ($L \subset \boldsymbol{R}^d$) we associate its *lower (upper) semicontinuous relaxation* $h_*(h^*): \bar{L} \to \tilde{\boldsymbol{R}} = \boldsymbol{R} \cup \{\pm\infty\}$ defined by

$$h_*(z)=\lim_{\varepsilon\downarrow 0} \inf\{h(y)\,;\,|z-y|<\varepsilon, y\in L\}, \quad z \in \bar{L}$$

and $h^*(z)=-(-h)_*(z)$. Let Ω be an open set in $\boldsymbol{R}^n$. For $T>0$ let W be a dense subset of $A=(0,T]\times\Omega\times\boldsymbol{R}\times\boldsymbol{R}^n\times\boldsymbol{S}^n$, where $\boldsymbol{S}^n$ denotes the space of $n\times n$ real symmetric matrices. Suppose that $F=F(t,x,r,p,X)$ is a real valued function defined in W. Since W is dense in A, F^* and $F_*: A\to\tilde{\boldsymbol{R}}$ are well-defined. Any function $u: Q\to\boldsymbol{R}$ (resp. $Q_0\to\boldsymbol{R}$) is called a *viscosity subsolution* of (1) in $Q=(0,T]\times\Omega$ (resp. $Q_0=(0,T)\times\Omega$) if $u^*<\infty$ on $\bar{Q}$ and if, whenever $\psi\in C^2(Q)$ (resp. $C^2(Q_0)$), $(t,x)\in Q$ (resp. Q_0) and $(u^*-\psi)(t,x)=\max_Q(u^*-\psi)$ (resp. $\max_{Q_0}(u^*-\psi)$) it holds that

$$\psi_t(t,x)+F_*(t,x,u^*(t,x),\nabla\psi(t,x),\nabla^2\psi(t,x))\le 0, \tag{2}$$

where $\psi_t=\partial\psi/\partial t$. We shall suppress the word viscocity. One can easily observe that u is a subsolution of (1) in Q (resp. Q_0) if and only if u is a subsolution of (1) in $(0,T]\times U(x)$ (resp. $(0,T)\times U(x)$) for all $x\in\Omega$, where $U(x)$ is an open ball centered at x in Ω.

*) On leave from Nankai Institute of Mathematics, Tianjin, China.
**) Department of Mathematics, Hokkaido University.
***) Department of Applied Science, Faculty of Engineering, Kyushu University.

2. Accessibility theorem. *Let k be a positive integer. Let $T>0$ and $y_{0i}\in\mathbf{R}^{n_i}$ $(1\le i\le k)$ and let Ω_i be an open set in $\mathbf{R}^{n_i}$ with $y_{0i}\in\Omega_i$. Let $A=A_i$ be as above with $\Omega=\Omega_i$ and W_i be a dense subset of A_i. Suppose that $F=F_i : W_i\to\mathbf{R}$ satisfies*

$$(3)\qquad \begin{aligned} &F_*(t,x,r,p,X)>-\infty \quad \text{for} \quad p\neq 0,\ r\in\mathbf{R},\ X\in\mathbf{S}^n\\ &F_*(t,x,r,0,O)>-\infty \quad \text{for} \quad r\in\mathbf{R}\end{aligned}$$

with $n=n_i$ and $t=T$ for all x near y_{0i} $(1\le i\le k)$. Let u_i be a subsolution of (1) with $F=F_i$ on $Q_i=(0,T]\times\Omega_i$. Then the function $w(t,z)=\sum_{i=1}^k u_i^*(t,z_i)$ is left accessible at (T,y_0), where $z=(z_1,\cdots,z_k)$, $z_i\in\Omega_i$ and $y_0=(y_{01},\cdots,y_{0k})$.

Example. The assumption (3) cannot be dropped even for $k=1$. Indeed, we observe that $u(t,x)=0$ for $t<T$ and $=1$ for $t=T$ is a subsolution of (1) with $F=F(p,X)=-(\text{trace } X)/|p|$ in $(0,T]\times\mathbf{R}^n$, since $F_*(0,O)=-\infty$ and F is degenerate elliptic, i.e. $F(p,X)\le F(p,Y)$ if $X\ge Y$ for usual ordering of $\mathbf{S}^n$. Clearly u is not left accessible at (T,y_0) for any $y_0\in\mathbf{R}^n$.

3. Lemma. *Let $\Phi(s,z)<+\infty$ be an upper semicontinuous (u.s.c) function on $Z=(\tau,T]^k\times D$, where D is a bounded open set in $\mathbf{R}^N$ and $\tau<T$. For $\delta>0$ let (t_δ,z_δ) be a maximizer of*

$$(4)\qquad \Phi_\delta(s,z)=\Phi(s,z)-\sum_{i=2}^k (s_1-s_i)^2/\delta,\quad s=(s_1,\cdots,s_k)$$

over $\bar Z$. Suppose that $\varphi(t,z)=\Phi(t,\cdots,t,z)$ attains its strict maximum over $[\tau,T]\times\bar D$ at (T,z_0), $z_0\in D$. Then each i-th component $t_{\delta i}$ of t_δ converges to T and z_δ converges to z_0 as $\delta\to 0$, where $1\le i\le k$. Moreover

$$(5)\qquad \lim_{\delta\to 0}\Phi_\delta(t_\delta,z_\delta)=\lim_{\delta\to 0}\Phi(t_\delta,z_\delta)=\varphi(T,z_0).$$

Proof. Since Φ_δ is maximized at (t_δ,z_δ), we see

$$\Phi(t_\delta,z_\delta)-\sum_{i=2}^k (t_{\delta 1}-t_{\delta i})^2/\delta\ge\Phi(T,\cdots,T,z_0)=\varphi(T,z_0).$$

This implies that $\sum_{i=2}^k (t_{\delta 1}-t_{\delta i})^2/\delta$ has an upper bound $\sup_Z\Phi-\varphi(T,z_0)$ independent of δ. In particular $t_{\delta 1}-t_{\delta i}\to 0$ as $\delta\to 0$ for $2\le i\le k$.

Suppose that $t_{\delta i}\to t_i'$ and $z_\delta\to z'$ by taking a subsequence $\delta=\delta_j\to 0$. Since $t_{\delta 1}-t_{\delta i}\to 0$, we see $t_i'=t_1'$ for $2\le i\le k$. From $\Phi_\delta\le\Phi$ it follows that

$$(6)\qquad \varphi(T,z_0)=\Phi_\delta(T,\cdots,T,z_0)\le\Phi_\delta(t_\delta,z_\delta)\le\Phi(t_\delta,z_\delta).$$

Letting $\delta_j\to 0$ yields $\varphi(T,z_0)\le\varphi(t_1',z')$ since Φ is u.s.c. This implies $t_1'=T$ and $z'=z_0$ since (T,z_0) is the strict maximizer of $\varphi(t,z)$. The inequality (6) now yields (5) since Φ is u.s.c. The proof is now complete by the compactness of $\bar Z$.

4. Proof of the accessibility theorem. We set

$$W(s,z)=W(s_1,\cdots,s_k,z)=\sum_{i=1}^k u_i^*(s_i,z_i),\quad s=(s_1,\cdots,s_k)$$

so that $W(t,\cdots,t,z)=w(t,z)$. Suppose that the conclusion were false. Then there would exist an open ball D_i in Ω_i centered at y_{0i} and $\varepsilon>0$ such that

$$a:=w(T,y_0)-\sup_U w(t,z)>0$$

with $U=(T-\varepsilon,T)\times\bar D$, $D=D_1\times D_2\times\cdots\times D_k$. We may assme that (3) holds for F_i at $t=T$ for all $x\in D_i$ by taking D_i smaller. We shall fix ε and D

and take K large so that $w(T,z)-\sum_{i=1}^{k}K|z_i-y_{0i}|^4$ attains a maximum M at $z=z_0\in D$ over $\bar{D}$. The function

$$w(T,z)-\sum_{i=1}^{k}P_i(z_i) \quad \text{with} \quad P_i(z_i)=K|z_i-y_{0i}|^4+|z_i-z_{0i}|^4$$

now attains a strict maximum M at $z_0=(z_{01},\cdots,z_{0k})$ over $\bar{D}$. We shall fix K.

We next introduce a function of t whose derivative at $t=T$ is very large. Let $\beta\in C^2(-\infty,0]$ satisfy $0\le\beta\le 1$ and $\beta(0)=\beta'(0)=1$. For $L>1$ we set $\beta_L(t)=a\beta(L(t-T))/2$. We now define Φ by

$$\Phi(s,z)=W(s,z)-\Xi(s,z) \quad \text{with} \quad \Xi(s,z)=\sum_{i=1}^{k}P_i(z_i)+\beta_L(s_1).$$

By the choice of β_L the function $\varphi(t,z)=\Phi(t,\cdots,t,z)$ would attain its strict maximum $M-a/2$ at (T,z_0) over $\bar{U}$. Let Φ_δ be as in (4), i.e.

$$\Phi_\delta(s,z)=W(s,z)-\Xi_\delta(s,z) \quad \text{with} \quad \Xi_\delta(s,z)=\Xi(s,z)+\sum_{i=2}^{k}(s_1-s_i)^2/\delta.$$

By Lemma 3 a maximizer (t_δ,z_δ) of Φ_δ over $[T-\varepsilon,T]^k\times\bar{D}$ would converge to $(T,\cdots,T,z_0)$ as $\delta\to 0$.

Since u_i is a subsolution of (1) in $Q_i'=(T-\varepsilon,T)\times D_i$ and since

$$u_i(t,x)-\Xi_\delta(t_{\delta 1},\cdots,t_{\delta i-1},t,t_{\delta i+1},\cdots,t_k,z_{\delta 1},\cdots,z_{\delta i-1},x,z_{\delta i+1},\cdots,z_{\delta k})$$

attains its maximum at $(t_{\delta i},z_{\delta i})$ over Q_i' (as a function of (t,x)), the inequality (2) yields

$$(7_i)\quad b_i(\delta)+f_i(\delta)\le 0 \quad \text{with} \quad f_i(\delta)=F_{i*}(t_{\delta i},z_{\delta i},u_i^*(t_{\delta i},z_{\delta i}),\nabla P_i(z_{\delta i}),\nabla^2 P(z_{\delta i})).$$

Here, $b_1(\delta)=(\beta_L)_t(t_{\delta 1})+2\sum_{j=2}^{k}(t_{\delta 1}-t_{\delta j})/\delta$ and $b_i(\delta)=-2(t_{\delta 1}-t_{\delta i})/\delta$ for $2\le i\le k$. Adding (7_i) from $i=1$ to k yields

$$(\beta_L)_t(t_{\delta 1})+\sum_{i=1}^{k}f_i(\delta)\le 0.$$

Since $t_{\delta i}\to T$ and $z_\delta\to z_0$, letting $\delta\to 0$ would yield

$$(8)\qquad La/2+\sum_{i=1}^{k}F_{i*}(T,z_{0i},u_i^*(T,z_{0i}),\nabla P_i(z_{0i}),\nabla^2 P_i(z_{0i}))\leqq 0$$

provided that

$$(9)\qquad \lim_{\delta\to 0}u_i^*(t_{\delta i},z_{\delta i})=u_i^*(T,z_{0i})\quad (1\leqq i\leqq k).$$

Since $\nabla P_i(z_{0i})=0$ implies $\nabla^2 P_i(z_{0i})=0$ and since z_0 is independent of L, the inequality (8) contradicts (3) for large L. Thus w is left accessible at (T,y_0).

It remains to prove (9). Since u_i^* is u.s.c. and Ξ is continuous, (5) yields (9).

5. Comparison theorem up to terminal time. *Suppose that* $F=F(t,r,p,X)$ *is continuous and degenerate elliptic on* $J_0=(0,T]\times \boldsymbol{R}\times(\boldsymbol{R}^n\setminus\{0\})\times\boldsymbol{S}^n$. *For each* $M>0$ *there is a constant* $c_0=c_0(n,T,M)$ *such that* $r\mapsto F(t,r,p,X)+c_0 r$ *is nondecreasing for all* $(t,r,p,X)\in J_0$ *with* $|r|\le M$. *Suppose that* $-\infty<F_*(t,r,0,O)=F^*(t,r,0,O)<\infty$. *Let* u *and* v *be respectively, sub- and supersolutions of* (1) *in* Q *with bounded* Ω. *If* $u^*\le v_*$ *on the parabolic boundary* $\partial_p Q=\{0\}\times\Omega\cup[0,T]\times\partial\Omega$, *then* $u^*\le v_*$ *on* Q.

This is proved in [1, Theorem 4.1] by extending Ishii's lemma ([8, Proposition IV. 1], [1, Proposition 3.1]) up to $t=T$ [1, Lemma 3.1]. It turns out that $u^*\le v_*$ for $t<T$ can be proved just by using original Ishii's lemma [1, Proposition 3.2] if we modify [1, Lemma 4.3]. To get $u^*\le v_*$ up to

$t=T$ we need to apply the Accessibility theorem. We just indicate how to alter the proofs of [1, Lemma 4.3 and Theorem 4.1].

In the statement of [1, Lemma 4.3] we should replace ψ by

$$\psi_\alpha(t,x,y)=\phi(x-y)+\alpha/(T-t)$$

for arbitrary fixed $\alpha>0$. One can carry out the proof of Case 1 with ψ_α by using [1, Proposition 3.2] since $\bar{t}<T$ and $\partial\psi_\alpha/\partial t>0$. In Case 2 we should replace $\tilde{\psi}$ and Φ_η by

$$\tilde{\psi}(t,x,y)=\psi_\alpha(t,x,y)+(\bar{t}-t)^2,$$
$$\Phi_\eta(t,x,y)=w(t,x,y)-\phi(x-y-\eta)-(\bar{t}-t)^2-\alpha/(T-t)$$

respectively. The Case 2a should be

'For some $\kappa>0$ there is $(t_\eta, x_\eta, y_\eta)\in Q_T$ with $x_\eta-y_\eta=\eta$ such that

$$\Phi_\eta(t_\eta,x_\eta,y_\eta)=\sup\{\Phi_\eta(t,x,y);\, x,y\in\Omega,\, |x-y|<\kappa,\, t\in(0,T]\}$$

for all $\eta\in \boldsymbol{R}^n$ with $|\eta|<\kappa$.'

In the proof for Case 2a we replace f by

$$f(\eta)=\sup\{w(t_\eta,x,y)-(\bar{t}-t_\eta)^2-\alpha/(T-t_\eta);\, x,y\in\Omega,\, x-y=\eta\}.$$

We argue in the same way as in the original proof and obtain

$$\sup\{w(t,x,y)-(\bar{t}-t)^2-\alpha/(T-t);\, |x-y|<\kappa,\, t\in(0,T]\}=w(\bar{t},\bar{x},\bar{x})-\alpha/(T-\bar{t})$$

in place of (4.9). Since $\bar{t}<T$, we apply [1, Proposition 3.2] to complete the proof for Case 2a. Again we should note $\partial\psi_\alpha/\partial t>0$ to get (4.12b). The remaining Case 2b can be treated parallely if we replace $Q_{\bar{t}}$ by Q_T. We note that the maximum of Φ_0 is not attained at $t\neq\bar{t}$ $(<T)$ because of the term $(\bar{t}-t)^2$ in $\tilde{\psi}$. We thus observe that [1, Lemma 4.3] with ψ_α holds for all $\alpha>0$.

In the proof of [1, Theorem 4.1] one should replace ψ by ψ_α. (All ϕ after the definition of w^ε were misprints of ψ so it should also be replaced by ψ_α.) We argue in the same way as in the original proof with ψ replaced by ψ_α and end up with $w^\varepsilon\le\psi_\alpha$ or

$$u(t,x)-v(t,y)\le a_\lambda(|x-y|^2+\delta)^{1/2}+b_\lambda+\alpha/(T-t) \quad \text{on} \quad Q_T.$$

Sending $\delta\to0$, $\alpha\to0$ and taking infimum for $\lambda\in A$ we obtain

$$(10) \qquad u(t,x)-v(t,y)\le m(|x-y|) \quad \text{for} \quad t<T,\, x,y\in\Omega,$$

where m is some modulus.

Since u and $-v$ are subsolutions of (1) with some F satisfying (3) on Q, the Accessibility theorem with $k=2$ implies that $u(t,x)-v(t,y)$ is left accessible at (T,x,y), $x,y\in\Omega$. We now conclude that (10) holds up to $t=T$ which yields $u^*\le v_*$ on Q.

Remark. In [5] the comparison theorem is extended to more general equations on arbitrary domains and the proof is simplified. However, since [5, Proposition 2.4] actually needs $t<T$ in the definition of α, the comparison [5, (2.2) and (4.2)] holds only for $t<T$ from the proof given there. Fortunately one applies the Accessibility theorem to get [5, (2.2) and (4.2)] up to $t=T$ so main results in [5] are correct as stated.

6. Ishii's lemma. We note that the conclusion of [1, Lemma 3.1] is correct if we assume that F and $-G(t,x,-r,-p,-X)$ satisfy (3) at $t=T$ for all $x\in\Omega$. Indeed, we may assume that U_T is bounded and that

(11) $$\Phi(t,x,y)=u(t,x)-v(t,y)-\phi(t,x,y)$$

attains its strict maximum over $\bar{U}_T$ as in [1, p. 763]. For $\alpha>0$ we introduce $\Phi_\alpha=\Phi-\phi_\alpha$ with $\phi_\alpha=\phi+\alpha/(T-t)$ which is different from that in [1, p. 763]. Let $(t_\alpha, x_\alpha, y_\alpha)$ be a maximizer of Φ_α on $\bar{U}_T$ so that $t_\alpha<T$. Suppose that $t_\alpha\to t'$, $x_\alpha\to x'$, $y_\alpha\to y'$ by taking a subsequence $\alpha=\alpha_j\to 0$. For $t<T$ we observe

$$\Phi(t,x,y)=\lim_{\alpha\to 0}\Phi_\alpha(t,x,y)\le\liminf_{\alpha\to 0}\Phi_\alpha(t_\alpha,x_\alpha,y_\alpha)\le\liminf_{\alpha\to 0}\Phi(t_\alpha,x_\alpha,y_\alpha)$$
$$\le\limsup_{\alpha\to 0}\Phi(t_\alpha,x_\alpha,y_\alpha)\le\Phi(t',x',y')\le\Phi(T,\bar{x},\bar{y})$$

since $\Phi_\alpha\le\Phi$ and Φ is u.s.c. Since $u(t,x)-v(t,y)$ is left accessible at $(T,\bar{x},\bar{y})$, this implies

(12) $$\lim_{\alpha\to 0}\Phi(t_\alpha,x_\alpha,y_\alpha)=\Phi(T,\bar{x},\bar{y}),\quad x'=\bar{x},\ y'=\bar{y}.$$

Since u and $-v$ are u.s.c., (12) yields

(13) $$\lim_{\alpha\to 0}u(t_\alpha,x_\alpha)=u(T,\bar{x}),\quad \lim_{\alpha\to 0}v(t_\alpha,y_\alpha)=v(T,\bar{y}).$$

We apply Ishii's lemma [1, Proposition 3.2] at $(t_\alpha, x_\alpha, y_\alpha)$ and send $\alpha\to 0$ to get the desired result [1, (3.4a) and (3.4b)] since $\partial\phi_\alpha/\partial t\ge\partial\phi/\partial t$.

The proof given in [1, p. 763] seems to be wrong because there may not exist the barrier m and the convergence in [1, p. 764, line 3] is not clear. However, as shown above [1, Lemma 3.1] is correct with extra assumptions of type (3) which causes no problem for the application in [1, Lemma 4.3].

By the way the proof of [1, Proposition 3.2] contains a minor technical error which can be easily fixed. In [1, p. 762, line 9–3 from below], the property that $F(t,x,r,p,X)$ and $G(t,x,r,p,X)$ are non *increasing* in r is used although it is not assumed in [1, Proposition 3.2]. This extra assumption is unnecessary because

(14) $$\lim_{j\to\infty}u^{\varepsilon_j}(t_j,x_j)=u(\bar{t},\bar{x}),\quad \lim_{j\to\infty}v_{\varepsilon_j}(t_j,y_j)=v(\bar{t},\bar{y})$$

with $t_j=t_{k_j}^{\varepsilon_j}$, $x_j=x_{k_j}^{\varepsilon_j}$, $\cdots$, where $\{\varepsilon_j\}$, $\{k_j\}$ are taken as in [1, p. 762, line 8]. We may assume $t_j\to\bar{t}$, $x_j\to\bar{x}$, $y_j\to\bar{y}$. As in the proof of (5), one can prove

$$\Phi(\bar{t},\bar{x},\bar{y})=\lim_{j\to\infty}\Phi_{\varepsilon_j,k_j}(t_j,x_j,y_j)$$

with $\Phi_{\varepsilon,k}(t,x,y)=\Phi(t,x,y)-l_k^\varepsilon t-p_k^\varepsilon\cdot x+q_k^\varepsilon\cdot y$ since $u\le u^\varepsilon$ and $v\ge v_\varepsilon$. This yields (14) since u and $-v$ are u.s.c. We thus conclude that [1, Proposition 3.2] is correct as it stated.

7. Extension theorem. *Suppose that u is a subsolution of* (1) *in Q_0. Then u^* is a subsolution of* (1) *in Q.*

The statement in [1, Lemma 5.7] is incorrect and should be replaced by this theorem. When u is continuous in Q this is proved in [9].

Proof. We may assume that Ω is bounded and that $u^*-\psi$ attains its strict maximum at (T,x_0) over Q with $\psi\in C^2(Q)$. Let (t_α,x_α) be a maximizer of $u^*-\psi_\alpha$ with $\psi_\alpha=\psi+\alpha/(T-t)$ for $\alpha>0$ so that $t_\alpha<T$. Since u^* is left accessible at (T,x_0) we observe $t_\alpha\to T$, $x_\alpha\to x_0$ and $u^*(T,x_0)=\lim_{\alpha\to 0}u^*(t_\alpha,x_\alpha)$ (cf. (12), (13)). Letting $\alpha\to 0$ in (2) with $\psi=\psi_\alpha$, $t=t_\alpha$ and $x=x_\alpha$ we get (2)

with ψ at (T, x_0) since $\partial\psi_a/\partial t > \partial\psi/\partial t$.

8. Localization lemma. (i) *Suppose that u is a subsolution of* (1) *in Q_0. Then for $T' < T$, u is a subsolution of* (1) *in $Q' = (0, T'] \times \Omega$.* (ii) *Suppose that v is a subsolution of* (1) *in Q. Then v is a subsolution of* (1) *in $(0, T') \times \Omega$ for $T' \leq T$.*

Proof. We may assume that Ω is bounded. Suppose that $u^* - \psi$ attains its strict maximum at (t_0, x_0) over Q' for $\psi \in C^2(Q')$. Extend ψ to $\psi \in C^2(Q)$ and set $\psi_\delta = \psi + g(t)/\delta$ with $\delta > 0$ where $g = 0$ for $t < t_0$ and $g = (t - t_0)^3$ for $t \geq t_0$, so that $g \in C^2(\boldsymbol{R})$. Let (t_δ, x_δ) be a maximizer of $u^* - \psi_\delta$ over $\overline{Q}$, so that $t_\delta \geq t_0$. Then

$$(15) \qquad (u^* - \psi)(t_0, x_0) = (u^* - \psi_\delta)(t_0, x_0) \leq (u^* - \psi_\delta)(t_\delta, x_\delta) \leq (u^* - \psi)(t_\delta, x_\delta)$$
$$\text{or} \quad g(t_\delta)/\delta + (u^* - \psi)(t_0, x_0) \leq (u^* - \psi)(t_\delta, x_\delta).$$

This implies that $g(t_\delta)/\delta$ is bounded as $\delta \to 0$. Since $t_\delta \geq t_0$ we now observe $t_\delta \to t_0$. Since u^* is u.s.c. and $t_\delta \to t_0$, sending $\delta \to 0$ in (15) yields $x_\delta \to x_0$. This argument also yields $\lim_{\delta \downarrow 0} u^*(t_\delta, x_\delta) = u^*(t_0, x_0)$. Sending δ to zero in (2) with $\psi = \psi_\delta$ at (t_δ, x_δ) yields (2) with ψ at (t_0, x_0) since $\partial\psi_\delta/\partial t \geq \partial\psi/\partial t$. This completes the proof of (i). The part (ii) can be proved easily.

The Accessibility theorem and the Localization lemma yield:

9. Corollary. *Suppose that u is a subsolution of* (1) *in Q. If F satisfies* (3) *for $(t, x) \in Q$, then u^* is left accessible at each $(t, x) \in Q$.*

10. Miscellaneous remarks. We note that [1, Theorem 5.6] can be proved without using [1, Lemma 5.7] and sup convolutions. A direct proof is found in [2]. We also note that one can correct the proof of [1, Theorem 5.6] given in [1] if we use Theorems 2 and 7; we need to assume (3) at $t = T$ for all $x \in \Omega$ in [1, Theorem 5.6].

By the way the equation [1, (1.6) or (5.14)] does not follow from [6]. The correct one is found in [4]. In [2] we actually need to assume a uniform bound of the gradient of T in (1.6) and that of ω in (2.13) to apply comparison results in [5].

References

[1] Y.-G. Chen, Y. Giga, and S. Goto: Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. J. Differential Geometry, **33**, 749–786 (1991) (announcement: Proc. Japan Acad., **65A**, 207–210 (1989)).

[2] ——: Analysis toward snow crystal growth. Functional Analysis and Related Topics (ed. S. Koshi). Sapporo 1990, World Scientific, pp. 43–57 (1991).

[3] M. G. Crandall and R. Newcomb: Viscosity solutions of Hamilton-Jacobi equations at the boundary. Proc. Amer. Math. Soc., **94**, 283–290 (1985).

[4] Y. Giga and S. Goto: Motion of hypersurfaces and geometric equations. J. Math. Soc. Japan, **44** (1992) (to appear).

[5] Y. Giga *et al.*: Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains. Indiana Univ. Math. J., **40**, 443–470 (1991).

[6] M. Gurtin: Towards a nonequilibrium thermodynamics of two-phase materials. Arch. Rational Mech. Anal., **100**, 275–312 (1988).

[7] H. Ishii: A boundary value problem of the Dirichlet type for Hamilton-Jacobi equations. Ann. Sc. Norm. Sup. Pisa, (IV) **16**, 105–135 (1989).

[8] H. Ishii and P.-L. Lions: Viscosity solutions of fully nonlinear second-order elliptic partial differential equations. J. Differential Equations, **83**, 26–78 (1990).

[9] D. Nunziante: Uniqueness of viscosity solutions of fully nonlinear second order parabolic equations with noncontinuous time-dependence. Differential Integral Equations, **3**, 77–91 (1990).