51. On Invariant Eigendistributions on $U(p, q)/(U(r)\times U(p-r, q))$

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1. Introduction. Let $X=G/H$ be a semisimple symmetric space, and $\mathcal{O}$ an H-invariant open subset of X. Let $\boldsymbol{D}(X)$ be the ring of invariant differential operators on X, and χ a character of $\boldsymbol{D}(X)$. A Schwartz distribution Θ on $\mathcal{O}$ is said to be an *invariant eigendistribution* (=*IED*) with an *infinitesimal character* χ, if (i) Θ is H-invariant, and (ii) $D\Theta=\chi(D)\Theta$ for all $D\in\boldsymbol{D}(X)$. We denote by $\mathcal{D}'_{\chi,H}(\mathcal{O})$ the set of all IED's on $\mathcal{O}$ with the infinitesimal character χ. Let X' be the subset of all regular semisimple elements in X. X' is H-invariant, open and dense. As is well-known, any $\Theta\in\mathcal{D}'_{\chi,H}(X')$ is a real analytic function. For any $\Theta\in\mathcal{D}'_{\chi,H}(X)$ we have clearly $\Theta|_{X'}\in\mathcal{D}'_{\chi,H}(X')$.

In the following, we take $X=U(p,q)/(U(r)\times U(p-r,q))$. Our aim is to determine IED's on X as explicitly as possible. For this end, we study the following problem, to which a corresponding problem for semisimple Lie groups was investigated in detail by Hirai [4]:

Problem. *Find a necessary and sufficient condition for an IED on X' to be extensible to an IED on X.*

In this article, we give a necessary condition in the case where the infinitesimal character is regular (cf. the last part of 2). It will be shown that our condition is also sufficient, when the infinitesimal character χ is "generic". We conjecture that this will hold even in the case where χ is not generic.

We briefly describe our method. We need first to know the following:

(i) The radial parts of invariant differential operators;

(ii) Invariant integrals, especially, their behavior around a singular semisimple element x in X.

The results on (i) were essentially given by Hoogenboom [5]. To investigate (ii), we consider the symmetric subspace $Z_G(x)/Z_H(x)$ of X defined by the centralizers of x in G and H respectively, and the invariant integrals on this subspace. From (i) and (ii), we can control, via Weyl's integral formula, the behavior of IED's around x, and hence we get our main results.

Results in the case of singular (i.e. non-regular) infinitesimal character will appear in our forthcoming paper.

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2. Preliminaries. Put $U(m,n)=\{g\in GL(m+n,\boldsymbol{C})\mid gI_{m,n}g^{*}=I_{m,n}\}$, where $I_{m,n}=\operatorname{diag}(\overbrace{1,\cdots,1}^{m},\overbrace{-1,\cdots,-1}^{n})$ and $g^{*}={}^{t}\bar{g}$. Let p, q and r be positive integers. We assume, for simplicity,

(※) $\qquad 2r\leqq p$ and $1\leqq r\leqq q$.

Let G denote the group $U(p,q)$. For $g\in G$, put $\sigma(g)=I_{r,p+q-r}gI_{r,p+q-r}$. Let H be the subgroup of G consisting of elements fixed by σ, then H is isomorphic to $U(r)\times U(p-r,q)$. Put $X=G/H$. We define an inclusion $\tilde{\sigma}$ of X into G by $\tilde{\sigma}(j)=g\sigma(g)^{-1}$ with $j=gH\in X$. By this map $\tilde{\sigma}$, X is identified with $\tilde{\sigma}(X)\subset G$. For $0\leqq l\leqq r$, we denote by $j_l(\theta_1,\cdots,\theta_l,t_{l+1},\cdots,t_r)$ the following matrix in $\tilde{\sigma}(X)$:

$$\left(\begin{array}{ccc|c}
\overbrace{\begin{matrix}\operatorname{ch}2t_r&&\\&\ddots&\\&&\operatorname{ch}2t_{l+1}\end{matrix}}^{r} & & \overbrace{}^{p-r} & \overbrace{\begin{matrix}&&\operatorname{sh}2t_r\\&{\cdot}^{\cdot^{\cdot}}&\\\operatorname{sh}2t_{l+1}&&\end{matrix}}^{q} \\
\begin{matrix}\cos2\theta_l&&\\&\ddots&\\&&\cos2\theta_1\\&&\sin2\theta_1\\&{\cdot}^{\cdot^{\cdot}}&\\\sin2\theta_l&&\end{matrix} & \begin{matrix}&&-\sin2\theta_l\\&{\cdot}^{\cdot^{\cdot}}&\\-\sin2\theta_1&&\\\cos2\theta_1&&\\&\ddots&\\&&\cos2\theta_l\end{matrix} & & \\
& & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & \\
\hline
& & & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\
\begin{matrix}&&\operatorname{sh}2t_{l+1}\\&{\cdot}^{\cdot^{\cdot}}&\\\operatorname{sh}2t_r&&\end{matrix} & & & \begin{matrix}\operatorname{ch}2t_{l+1}&&\\&\ddots&\\&&\operatorname{ch}2t_r\end{matrix}
\end{array}\right).$$

Put $J_l=\{j_l(\theta_1,\cdots,\theta_l,t_{l+1},\cdots,t_r)\mid\theta_1,\cdots,\theta_l,t_{l+1},\cdots,t_r\in\boldsymbol{R}\}$. Then $J_0,\cdots,J_r$ form a complete system of representatives of H-conjugacy classes of Cartan subspaces, analogues of Cartan subgroups. (J_0 is split and J_r is compact.) So the rank of X (i.e. the dimension of J_l) is equal to r. The Weyl group $W(J_l)=N_H(J_l)/Z_H(J_l)$ of J_l is isomorphic to the semidirect product $(\boldsymbol{Z}_2)^r\rtimes(\mathfrak{S}_l\times\mathfrak{S}_{r-l})$, where we denote by $\mathfrak{S}_n$ the symmetric group of degree n. Putting $J'_l=J_l\cap X'$, we get $X'=\coprod_{l=0}^{r}H\cdot J'_l$. For an element $j_l(\theta_1,\cdots,\theta_l,t_{l+1},\cdots,t_r)$ in J_l, we consider the r-tuple of real numbers $(\tau_1,\cdots,\tau_r)$ given by

$$\tau_i=\cos^2\theta_i\quad(1\leqq i\leqq l),\qquad \tau_i=\operatorname{ch}^2 t_i\quad(l+1\leqq i\leqq r).$$

By the mapping $j\mapsto(\tau_1,\cdots,\tau_r)$, J_l is mapped onto the set

$(*)_l \qquad \{(\tau_1,\cdots,\tau_r)\mid 0\leqq\tau_i\leqq 1\ (1\leqq i\leqq l),\ 1\leqq\tau_i\ (l+1\leqq i\leqq r)\}.$

Note that j belongs to J_l' if and only if $\tau_i\neq 0, 1$ $(1\leqq i\leqq r)$, $\tau_i\neq\tau_j$ $(1\leqq i<j\leqq r)$. Corresponding to the action of the Weyl group, the group $\mathfrak{S}_l\times\mathfrak{S}_{r-l}$ acts on the set $(*)_l$ through the permutation of the variables. Collecting over $0\leqq l\leqq r$, we see that a function on the set

$$\{(\tau_1,\cdots,\tau_r)\,|\,0<\tau_1<\tau_2<\cdots<\tau_r,\ \tau_i\neq 1\ (i=1,\cdots,r)\}$$

is uniquely extensible to an H-invariant function on X'.

Putting $\mu=p+q-2r$, we define an H-invariant open subset X_1 of X as follows:

$$X_1=\{j\in X\,|\,\text{the matrix } \tilde{\sigma}(j) \text{ has eigenvalue 1 with the multiplicity at most } \mu+2\}.$$

Then j in $\bigcup_{l=0}^r J_l$ belongs to X_1 if and only if $\tau_i=1$ for at most one i in the r-tuple $(\tau_1,\cdots,\tau_r)$ corresponding to j. Put $\omega=\prod_{1\leqq j<i\leqq r}(\tau_i-\tau_j)$ and

$$L_i=4\tau_i(\tau_i-1)\frac{\partial^2}{\partial\tau_i^2}+4\{(\mu+2)\tau_i-1\}\frac{\partial}{\partial\tau_i}\quad(i=1,\cdots r),$$

$$\mathcal{S}=\{\omega^{-1}S(L_1,\cdots,L_r)\omega\,|\,S \text{ is a symmetric polynomial with } r\text{-variables}\}.$$

Due to Hoogenboom [5], we have the isomorphism Φ of $\boldsymbol{D}(X)$ onto $\mathcal{S}$ satisfying $\Phi(D)|_{J_l'}(f|_{J_l'})=(Df)|_{J_l'}$ for any $D\in\boldsymbol{D}(X)$, $f\in C^\infty(X)$ and $l=0,1,\cdots,r$. We note that $\Phi(D)|_{J_l'}$ is the radial part of $D\in\boldsymbol{D}(X)$ on J_l'.

In view of this fact, let D_k denote the element of $\boldsymbol{D}(X)$ defined by the condition

$$\Phi(D_k)=\omega^{-1}(L_1^k+L_2^k+\cdots+L_r^k)\omega,$$

then $D_1, D_2, \cdots, D_r$ form a system of free generators of the commutative algebra $\boldsymbol{D}(X)$. Hence we have the bijection $\chi\mapsto(\lambda_1,\cdots,\lambda_r)$ of the set of the characters of $\boldsymbol{D}(X)$ onto the set of non-ordered r-tuples of complex numbers through

$$\chi(D_k)=\lambda_1^k+\lambda_2^k+\cdots+\lambda_r^k\quad(k=1,2,\cdots,r).$$

This character χ is denoted by $\chi_{\lambda_1,\dots,\lambda_r}$.

In case $\lambda_i\neq\lambda_j$ $(i\neq j)$, $\chi=\chi_{\lambda_1,\dots,\lambda_r}$ is called *regular*. Otherwise, χ is called *singular*.

3. Main results. Throughout this section, we assume that $\chi=\chi_{\lambda_1,\dots,\lambda_r}$ is regular, that is, $\lambda_1,\cdots,\lambda_r$ are distinct complex numbers. We set

$\rho=\mu+1=p+q-2r+1$, $\lambda(s)=s^2-\rho^2$ and

$$\Lambda_d=\{\lambda(s)\,|\,s=\pm(\rho+2i), i=0\ 1,2,\cdots\}=\{4i(i+\mu+1)\,|\,i=0,1,2,\cdots\},$$

$$\Lambda_s=\{\lambda(s)\,|\,s=\pm(\rho+2i), i=-1,-2,\cdots,-\mu\}=\{4i(i+\mu+1)\,|\,i=-1,-2,\cdots,-\mu\},$$

and put

$$L=4\tau(\tau-1)\frac{d^2}{d\tau^2}+4\{(\mu+2)\tau-1\}\frac{d}{d\tau}.$$

Let $F(\tau,\lambda)$ denote the real analytic function in $\tau>0$ satisfying the differential equation $(L-\lambda)F=0$ and $F(1,\lambda)=1$. The function $G(\tau,\lambda)=F(1-\tau,\lambda)$ is a hypergeometeric function. We note that $F(\tau,\lambda)$ is real analytic at $\tau=0$ if and only if λ belongs to Λ_d.

Put $\mathfrak{S}_{r,l}=\{k\in\mathfrak{S}_r\,|\,\lambda_{k(i)}\in\Lambda_d\ (1\leqq i\leqq l)\}$. As a complete system R_l of representatives of $\mathfrak{S}_{r,l}/(\mathfrak{S}_l\times\mathfrak{S}_{r-l})$, we take

$$R_l=\{k\in\mathfrak{S}_{r,l}\mid k(1)<k(2)<\cdots<k(l),\ k(l+1)<k(l+2)<\cdots<k(r)\}.$$

We note that $R_0=\{id\}$. For any parameter $\nu_1,\cdots,\nu_s\in \boldsymbol{C}$, put

$$D_{\nu_1,\dots,\nu_s}(x_1,\cdots,x_s)=\det\begin{pmatrix}F(x_1,\nu_1)\cdots\cdots F(x_s,\nu_1)\\ \vdots & \vdots\\ F(x_1,\nu_s)\cdots\cdots F(x_s,\nu_s)\end{pmatrix}.$$

Let $\prod$ be a function on $\bigcup_{l=0}^{r} J'_l$, and let $\prod_l$ denote the restricton of $\prod$ to J'_l. We consider the following three conditions with respect to $\prod=(\prod_l)_{l=0,1,\dots,r}$.

(1) For any $l\in\{0,1,2,\cdots,r\}$, the function $\omega\prod_l(\tau_1,\cdots,\tau_r)$ is a linear combination of

$$D_{\lambda_{k(1)},\dots,\lambda_{k(l)}}(\tau_1,\cdots,\tau_l)D_{\lambda_{k(l+1)},\dots,\lambda_{k(r)}}(\tau_{l+1},\cdots,\tau_r)\text{'s}$$

for k in R_l. ($\omega\prod_l\equiv 0$, in case $R_l=\varnothing$ i.e. $l>{}^{\#}[\{\lambda_1,\lambda_2,\cdots,\lambda_r\}\cap\Lambda_d]$.)

(2) There exists a $\Theta\in\mathcal{D}'_{\chi,H}(X)$ such that $\prod$ is the restriction of Θ to $\bigcup_{l=0}^{r} J'_l$.

(3) There exists a $\Theta\in\mathcal{D}'_{\chi,H}(X_1)$ such that $\prod$ is the restriction of Θ to $\bigcup_{l=0}^{r} J'_l$.

It is clear that condition (2) implies condition (3).

We describe our results, dividing into two cases, according as (A): $\Lambda_s\cap\{\lambda_1,\lambda_2,\cdots,\lambda_r\}\neq\varnothing$, or (B): $\Lambda_s\cap\{\lambda_1,\lambda_2,\cdots,\lambda_r\}=\varnothing$.

Theorem 1. *In case* (A), *condition* (3) *implies* $\Pi\equiv 0$.

By Theorem 1, if $\lambda_i\in\Lambda_s$ for some $1\leqq i\leqq r$, the support of any IED on X with $\chi_{\lambda_1,\dots,\lambda_r}$ is contained in the singular set $X-X'$.

Theorem 2. *In case* (B), *condition* (3) *is equivalent to condition* (1).

From Theorem 2, we get immediately the following:

Theorem 3. *In case* (B), *condition* (2) *implies condition* (1).

At present, we conjecture that the converse of Theorem 3 is also valid. That is, we propose

Conjecture. *In case* (B), *condition* (2) *is equivalent to condition* (1).

Remark. Assume ${}^{\#}[\{\lambda_1,\lambda_2,\cdots,\lambda_r\}\cap\Lambda_d]=0$. Due to Theorem 3, if a function $\prod=(\prod_l)_{l=0,\dots,r}$ on $\bigcup_{l=0}^{r} J'_l$ satisfies codition (2) for $\chi=\chi_{\lambda_1,\dots,\lambda_r}$, then we have

$$(1')\qquad \begin{cases}\prod_l=0 & \text{for } 1\leqq l\leqq r, \text{ and}\\ \omega\prod_0=c\det(F(\tau_i,\lambda_j))_{1\leqq i,j\leqq r} & \text{for some } c\in\boldsymbol{C}.\end{cases}$$

In particular, we have $\dim\{\Theta|_{X'}\mid\Theta\in\mathcal{D}'_{\chi,H}(X)\}\leqq 1$. On the other hand, for a "generic" χ, there exists a $\Theta\in\mathcal{D}'_{\chi,H}(X)$ satisfying the condition $\Theta|_{J'_0}\not\equiv 0$, according to the oral communication by T. Oshima. From these facts, we get the following assertion which supports our conjecture.

Let $\chi=\chi_{\lambda_1,\dots,\lambda_r}$ *be regular and "generic". Then condition* (2) *is equivalent to condition* (1′).

The details of this note will appear elsewhere.

References

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