42. On the Uniform Attractivity of Solutions of Ordinary Differential Equations by Two Lyapunov Functions

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(Communicated by Kunihiko KODAIRA, M. J. A., May 13, 1991)

1. Introduction. Consider the ordinary differential equation

(1) $$x'=f(t,x)\ (f(t,0)=0 \text{ for all } t\in \boldsymbol{R}_+ :=[0,\infty)),$$

where $f:\boldsymbol{R}_+\times \boldsymbol{R}^n\to \boldsymbol{R}^n$ is continuous.

K. Murakami and M. Yamamoto [10] have given sufficient conditions for the global attractivity and equi-attractivity of the zero solution of (1) based on Lyapunov functions with negative semidefinite derivatives. Nowadays such Lyapunov functions have been often used to investigate the asymptotic behaviour of solutions [1–16].

As is well-known, the *uniform* stability properties are of practical importance, e.g. if f satisfies a Lipschitz condition in x uniformly with respect to t, then the uniform attractivity together with uniform stability imply the total stability of the zero solution (see [12], Chapter II, Theorem 4.5).

In this paper we show that, after slightly strenthening one of them, the conditions in Murakami's and Yamamoto's theorem of the global equi-attractivity (Theorem 1 in [10]) imply also the global *uniform* attractivity. In our second theorem we can guarantee the global *equi*-attractivity under essentially weaker conditions than those of Murakami's and Yamamoto's theorem on the global attractivity (Theorem 2 in [10]).

2. Notations and definitions. We use the n-dimensional real space $\boldsymbol{R}^n$ with the Euclidean norm $|\cdot|$. If $x\in \boldsymbol{R}^n$, $F\subset \boldsymbol{R}^n$, we define the distance between x and F by $d(x,F):=\inf\{|x-y|: y\in F\}$. $B(\rho)$ and $\bar{B}(\rho)$ denote the ball of radius $\rho>0$ around the origin and its closure, respectively.

Definition 1. A measurable function $\varphi: \boldsymbol{R}_+\to \boldsymbol{R}_+$ is said to be *integrally positive* if $\int_I \varphi(s)\,ds=\infty$ for every set

(2) $$I=\bigcup_{k=1}^{\infty}[\alpha_k,\beta_k],\qquad \beta_k-\alpha_k\geq\delta>0\ (k\in N).$$

If, in addition to (2), the inequalities $\Delta\geq\beta_k-\alpha_k$ $(k\in N)$ are also required of I, then φ is called *weakly integrally positive* [3].

It is easy to see that φ is integrally positive if and only if

(3) $$\liminf_{t\to\infty}\int_t^{t+\gamma}\varphi(s)\,ds>0$$

for every $\gamma>0$. Moreover, if φ is integrally positive, then it is weakly integrally positive, but the converse is not rue (e.g. $\varphi(t):=(1+t)^{-1}$). One of the purposes of this paper is to emphasize that the weak integral positivity

*) This research was supported by the Hungarian Foundation for Scientific Research with grant number 6032/6319.

can often substitute for the integral positivity to guarantee non-uniform stability properties [3, 4].

In the assumption on the derivative of the Lyapunov function we will use a continuous function $V^*: \boldsymbol{R}^n \to \boldsymbol{R}_+$. Following Murakami and Yamamoto, we denote by $E(V^*=0)$ the zero set of V^*, and introduce the following notations:

$$S(\rho):=\{x \in \boldsymbol{R}^n : d(x, E(V^*=0))<\rho\} \qquad (\rho>0)$$
$$A(\rho_1, \rho_2):=B(\rho_2)\setminus\bar{B}(\rho_1);\ H(\rho_1, \rho_2):=S(\rho_2)\setminus\bar{S}(\rho_1) \qquad (0<\rho_1<\rho_2).$$

Definition 2. A function $Z: \boldsymbol{R}_+ \times \boldsymbol{R}^n \to \boldsymbol{R}$ is said to be *strictly non-zero in the set* $E(V^*=0)$ if for every γ, Γ $(0<\gamma<\Gamma)$ there are a number $r(\gamma, \Gamma)>0$ and a measurable function $\xi_{\gamma,\Gamma}: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ with

$$\lim_{R\to\infty} \int_t^{t+R} \xi_{\gamma,\Gamma}(s)\,ds = \infty \qquad (t \in \boldsymbol{R}_+), \tag{4}$$

and such that Z does not change its sign and $|Z(t,x)| \geq \xi_{\gamma,\Gamma}(t)$ on the set $\boldsymbol{R}_+ \times A(\gamma, \Gamma) \cap S(r(\gamma, \Gamma))$.

If (4) is satisfied uniformly with respect to $t \in \boldsymbol{R}_+$, then Z is called *uniformly strictly non-zero in* $E(V^*=0)$.

If $\xi_{\gamma,\Gamma}$ is integrally positive (respectively, $\xi_{\gamma,\Gamma}(t)=$const.), then Z is called *definitely non-zero in the integral sense* (respectively, *definitely non-zero*) *in* $E(V^*=0)$.

For $t \in \boldsymbol{R}_+$, $x_0 \in \boldsymbol{R}^n$ we denote by $x(t; t_0, x_0)$ any solution of (1) with $x(t_0; t_0, x_0)=x_0$.

Definition 3. The zero solution of (1) is said to be *globally attractive* if $|x(t; t_0, x_0)| \to 0$ as $t \to \infty$ for all $t \in \boldsymbol{R}_+$, $x_0 \in \boldsymbol{R}^n$. It is *globally equi-attractive* if the convergence is uniform with respect to $x_0 \in B(\sigma)$ for every $\sigma>0$. If the convergence is uniform with respect to $t_0 \in \boldsymbol{R}_+$, too, then the zero solution is called *globally uniformly attractive*.

We denote by $C_0(x)$ the family of continuous functions $V: \boldsymbol{R}_+ \times \boldsymbol{R}^n \to \boldsymbol{R}$ which satisfy a Lipschitz conditions with respect to x. For a $V \in C_0(x)$ we define the derivative of V with respect to (1) (see [15]) by

$$\limsup_{h\to 0+} \{(1/h)[V(t+h, x+hf(t,x)) - V(t,x)]\}.$$

$\mathcal{K}$ denotes the class of continuous functions $a: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ which are strictly increasing and vanishing at zero.

The results. Theorem 1. *Suppose that there are functions* $V, W \in C_0(x)$ *satisfying the following conditions in the set* $\boldsymbol{R}_+ \times \boldsymbol{R}^n$:

1) $a(|x|) \leq V(t,x) \leq b(|x|)$, *where* $a, b \in \mathcal{K}$, *and* $a(r) \to \infty$ *as* $r \to \infty$;

2) $V'(t,x) \leq -\varphi(t)V^*(x) + \psi(t)$, *where* $V^*: \boldsymbol{R}^n \to \boldsymbol{R}_+$ *is continuous,* φ *is integrally positive, and* $\psi: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *is integrable over* $\boldsymbol{R}_+$;

3) *there exists an* L *such that* $|W(t,x)| < L$;

4) $W'(t,x)$ *is uniformly strictly non-zero in the set* $E(V^*=0)$;

5) *for any compact set* $M \subset \boldsymbol{R}^n$ *and for any locally absolutely continuous function* $u: \boldsymbol{R}_+ \to M$, *the function* $\int_0^t f(s, u(s))\,ds$ *is uniformly continuous on* $\boldsymbol{R}_+$.

Then the zero solution of (1) *is uniformly globally attractive.*

Proof. It can be devided into eight steps:

1° The zero solution is eventually uniformly stable [15], i.e. for every $\varepsilon>0$ there are $\zeta_1(\varepsilon)$, $\delta(\varepsilon)>0$ such that $[t\geq t_0\geq\zeta_1,\ |x_0|<\delta(\varepsilon)]$ imply $|x(t;t_0,x_0)|<\varepsilon$.

In fact, consider the function $U(t,x):=V(t,x)+\int_t^\infty\psi$. By condition 2), U is nonincreasing along any solution x; therefore, we have the inequality

$$a(|x(t)|)+\int_t^\infty\psi\leq U(t,x(t))\leq U(t_0,x_0)\leq b(|x_0|)+\int_{t_0}^\infty\psi$$

for all $t\geq t_0$. Let $\zeta_1(\varepsilon)$ be chosen so that $\int_{\zeta_1}^\infty\psi<a(\varepsilon)/2$, and let $\delta(\varepsilon):=b^{-1}(a(\varepsilon)/2)$. If $t_0\geq\zeta_1$ and $|x_0|<\delta$, then $a(|x(t)|)<a(\varepsilon)$ and, consequently, $|x(t)|<\varepsilon$ for all $t\geq t_0$.

2° The solutions are uniformly bounded [15], i.e. for every $\sigma>0$ there is a $\Gamma(\sigma)$ such that $[t\geq t_0\geq 0,\ |x_0|<\sigma]$ imply $|x(t;t_0,x_0)|\geq\Gamma(\sigma)$.

In fact, for any solution x with $|x_0|<\sigma$ we obtain $a(|x(t)|)\leq U(t,x(t))\leq b(\sigma)+\int_0^\infty\psi$, so the choice $\Gamma(\sigma):=a^{-1}(b(\sigma)+\int_0^\infty\psi)$ is suitable.

3° In order to prove the assertion of the theorem we have to show that for every $\sigma>0$, $\eta>0$ there is a $T(\sigma,\eta)$ such that if $|x_0|<\sigma$, then $|x(t;t_0,x_0)|<\eta$ for all $t_0\in\boldsymbol{R}_+$, $t\geq t_0+T(\sigma,\eta)$. In the consequence of the eventual uniform stability of the zero solution (see 1°), to this end it is enough to prove the existence of $T(\sigma,\eta)$ and $t_*\in[t_0,t_0+T(\sigma,\eta)]$ with the properties $t_*\geq\zeta_1(\eta)$, $|x(t_*)|<\delta(\eta)$.

Let $\sigma>0$, $\eta>0$ be fixed. Suppose that $t_0\geq\zeta_1(\eta)$, $|x_0|<\sigma$ and $|x(t;t_0,x_0)|\geq\delta(\eta)=:\gamma(\eta)=\gamma$ for all $t\in[t_0,t_0+T_*]$ i.e. $x(t)\in A(\gamma(\eta),\Gamma(\sigma))$ on the interval $[t_0,t_0+T_*]$. Consider the number $r=r(\gamma(\eta),\Gamma(\sigma))$ and the function $\xi=\xi_{\gamma(\eta),\Gamma(\sigma)}$ corresponding to the function $W'(t,x)$ in the sense of condition 4) and Definition 2.

4° There exists an upper bound $T_1=T_1(\sigma,\eta)$ for the length of any interval of time $[\alpha,\beta]\subset[t_0,t_0+T_*]$ while the point $x(t)=x(t;t_0,x_0)$ can be staying in $S(r)$.

In fact, by (4) there is a $T_1=T_1(\sigma,\eta)$ such that

$$\int_t^{t+T_1}\xi_{\gamma(\eta),\Gamma(\sigma)}(s)\,ds>2L\qquad\text{for all } t\in\boldsymbol{R}_+.$$

Since

$$2L\geq|W(\alpha,x(a))-W(\beta,x(\beta))|\geq\left|\int_\alpha^\beta W'(t,x(t))\,dt\right|\geq\int_\alpha^\beta\xi(s)\,ds,$$

the inequality $\beta-a<T_1$ has to be satisfied.

5° There exists an upper bound $T_2=T_2(\sigma,\eta)$ for the length of any interval of time $[\alpha,\beta]\subset[t_0,t_0+T_*]$ of staying out of $S(r/2)$.

Let

$$m_1=m_1(\sigma,\eta):=\min\{V^*(x):\ x\in\bar{B}(\Gamma)\backslash S(r/2)\}.$$

Then

$$b(\sigma)+\int_0^\infty\psi\geq U(\alpha)-U(\beta)\geq\int_\alpha^\beta\varphi(t)V^*(x(t))\,dt\geq m_1\int_\alpha^\beta\varphi.$$

By property (3), the existence of $T_2(\sigma,\eta)$ follows from the integral positivity of φ.

6° There exists a positive lower bound $T_3=T_3(\sigma,\eta)$ for the transit time while $x(t)$ is crossing $H(r/2,r)$.

If $x(\alpha)\in\bar{S}(r/2)$ and $x(\beta)\notin S(r)$, then $r/2\leq|x(\alpha)-x(\beta)|=\left|\int_\alpha^\beta f(t,x(t))\,dt\right|$. By condition 5), there is a $T_3=T_3(\sigma,\eta)$ such that $|\alpha-\beta|<T_3$ implies $|x(\alpha)-x(\beta)|<r/2$. This T_3 is suitable for the desired lower bound.

7° There is an upper bound $M=M(\sigma,\eta)\in\boldsymbol{N}$ for the number of crossing $H(r/2,r)$.

In fact, introducing the notation

$$m_2=m_2(\sigma,\eta):=(1/2)\liminf_{t\to\infty}\int_t^{t+T_3(\sigma,\eta)}\varphi(s)\,ds$$

we have

$$\int_t^{t+T_3}\varphi\geq 3m_2/2,\quad \int_t^\infty\psi<m_1m_2/2$$

for $t\geq\zeta_2$ with some sufficiently large $\zeta_2=\zeta_2(\sigma,\eta)\geq\zeta_1(\sigma,\eta)$.

Since $U'(t,x(t))\leq-\varphi(t)V^*(x(t))$, the function $U(t,x(t))$ decreases at least by m_1m_2 while $x(t)$ is crossing $H(r/2,r)$ after ζ_2. But $U(t,x(t))$ is decreasing and $0\leq U(t,x(t))\leq b(\sigma)+\int_0^\infty\psi$ in the whole interval $[t_0,\infty)$, so $x(t)$ can cross $H(r/2,r)$ in $[\zeta_2,\infty)\cap[t_0,t_0+T_*]$ at most

$$M=M(\sigma,\eta):=\left[\left(b(\sigma)+\int_0^\infty\psi\right)\Big/m_1m_2\right]+2$$

times, where $[s]$ denotes the integral part of $s\in\boldsymbol{R}$.

8° Define now the number

$$T(\sigma,\eta):=\zeta_2(\sigma,\eta)+(M(\sigma,\eta)+1)(T_1(\sigma,\eta)+T_2(\sigma,\eta)).$$

It is easy to see that $T_*<T(\sigma,\eta)$, i.e. $x(t)$ cannot remain in the annulus $A(\gamma(\eta),\Gamma(\sigma))$ longer than $T(\sigma,\eta)$. This means that there is a $t_*\in[t_0,t_0+T(\sigma,\eta)]$ with $|x(t_*)|<\gamma(\eta)$, and, by the definition of $\gamma(\eta)$, $|x(t)|<\eta$ for all $t\geq t_0+T(\sigma,\eta)$.

The proof is complete.

The following theorem can be proved similarly.

Theorem 2. *Suppose that there are functions $V,W\in C_0(x)$ satisfying the following conditions in the set $\boldsymbol{R}_+\times\boldsymbol{R}^n$:*

1) *$a(|x|)\leq V(t,x)$, where $a\in\mathcal{K}$ and $a(r)\to\infty$ as $r\to\infty$;*

2) *$V'(t,x)\leq-\varphi(t)V^*(x)+\psi(t)$, where $V^*:\boldsymbol{R}^n\to\boldsymbol{R}_+$ is continuous, φ is weakly integrally positive, and $\psi:\boldsymbol{R}_+\to\boldsymbol{R}_+$ is integrable over $\boldsymbol{R}_+$;*

3) *for every γ,Γ $(0<\gamma<\Gamma)$ there is a function $c=c_{\gamma,\Gamma}\in\mathcal{K}$ such that*

$$|W(t,x)|\leq c(d(x,E(V^*=0)))\qquad(t\in\boldsymbol{R}_+,\ x\in A(\gamma,\Gamma));$$

4) *$W'(t,x)$ is definitely non-zero in the integral sense in the set $E(V^*=0)$;*

5) *for any compact set $M\subset\boldsymbol{R}^n$ and for any locally absolutely continuous function $u:\boldsymbol{R}_+\to M$, the function $\int_0^t f(s,u(s))\,ds$ is uniformly continuous in $\boldsymbol{R}_+$.*

Then the zero solution of (1) *is globally equi-attractive.*

4. **Remarks.** 1. In Theorem 1 of [10] the function $W'(t, x)$ was supposed to be only strictly non-zero in the set $E(V^*=0)$, but only global equi-attractivity was proved. It is worth noticing that this result can be deduced also from localization theorems [2, 3, 8, 12].

In fact, from Corollary 3.2 in [2] it follows that $x(t)\rightarrow E(V^*=0)$ as $t\rightarrow\infty$ for every solution x. On the other hand, since W is bounded and W' is strictly non-zero, for every γ, Γ $(0<\gamma<\Gamma)$ there is an $r=r(\gamma,\Gamma)>0$ such that the point $x(t)$ cannot remain in the set $A(\gamma,\Gamma)\cap S(r)$ for a long time. These facts yield $x(t)\rightarrow 0$ $(t\rightarrow\infty)$ due to the eventual uniform stability of the zero solution (see step 1° in the proof of Theorem 1).

2. If in condition 2) in Theorem 1 we require only the weak integral positivity of φ instead of the integral positivity, then we can guarantee only global equi-attractivity.

3. If $\psi(t)\equiv 0$ in condition 2) in Theorem 1 (Theorem 2), then the zero solution of (1) is globally uniformly asymptotically stable (globally equi-asymptotically stable, respectively) (as for the definitions see e.g. [6]).

4. In [10], instead of our 5), the following condition was required: for any compact set $M\subset \boldsymbol{R}^n$ there are a number N and a function r such that $\int_t^{t+1} r\rightarrow 0$ $(t\rightarrow\infty)$ and $|f(t,x)|\leq N+r(t)$ for all $t\in \boldsymbol{R}_+$, $x\in M$. It can be seen that this condition implies our condition 5), but the converse is not true.

5. Our Theorem 2 improves and sharpens that of [10]: in [10] φ was integrally positive and W' was definitely non-zero in $E(V^*=0)$; nevertheless, only the global attractivity was guaranteed.

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