34. *Weinstein Conjecture and a Theory of Infinite Dimensional Cycles*

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Introduction. Let (M, ω) be a contact manifold of dimension $2n+1$. Then there exists on M a vector field ξ, called a characteristic field (or Reeb field) such that

$$d\omega(\cdot, \xi)\equiv 0,$$
$$\omega(\xi)\equiv 1.$$

If M is an imbedded star-shaped sphere in $\boldsymbol{R}^{2n+2}$, and if f is a smooth function on $\boldsymbol{R}^{2n+2}$ such that $M=f^{-1}(k)$ for some $k\in \boldsymbol{R}$ and df is nowhere zero on M, then ξ is a Hamiltonian vector field of f with respect to the canonical symplectic structure Ω on $\boldsymbol{R}^{2n+2}$ (after a normalization). A. Weinstein [5] and P. Rabinowitz [4] showed there exists at least one closed orbit of ξ for any star-shaped sphere. In view of this result, the existence of closed orbits of ξ for any compact contact manifolds was conjectured by A. Weinstein.

For compact hypersurfaces of contact type in $\boldsymbol{R}^{2n+2}$, the conjecture was solved affirmatively by Viterbo [6]. His result was extended by Floer, Hoffer and Viterbo [2] for compact hypersurfaces of contact type in $\boldsymbol{C}^l\times P$, here (P, Ω) is a compact symplectic manifold, $l>0$ and Ω is supposed to vanish on $\pi_2(P)$.

This problem has the following variational aspect. Closed orbits of ξ coincide with the critical points of the following variational problem:

$$L(c)=\int \omega(\dot{c})ds$$
$$c\in C^1(S^1, M)$$

A neck of solving the conjecture for a general case lies in a break-down of the so calld Palais-Smale condition. This leads us to the notion of *critical points at infinity*, which are defined to be the set of *limit points* of sequences c_i such that the action of c_i tends to zero. In this paper we discuss this failure of the Palais-Smale condition and identify these critical points at infinity, using a theory of infinite dimensional cycles.

We define in the next section a family of operators $P=\{P_c\}$ parametrized by a free loop space $C^1(S^1, M)$. We derive from this family of operators a number of infinite dimensional cycles in the space $C^1(S^1, M)$. A general theory of infinite dimensional cycles associated to operators was studied in [3], to which we refer for notations of cycles. Among these cycles, our interest lies in a solution cycle $\kappa^{1,1}(P)$.

We suppose that the Stiefel-Whitney classes $w_{2n}(M)$, $w_{2n-1}(M)$ are equal to zero. Let v be a non-zero vector field in $ker(\omega)$. Then we have:

Theorem. (i) *The critical points at infinity on* $\kappa^{1,1}(P)$ *are piecewise smooth curves, broken at points* $\{p_i\}$, *such that* (1) *each segment from* p_i *to* p_{i+1} *is an orbit of* ξ *or* v. (2) p_i *is conjugate to* p_{i+1}.

(ii) *The cohomology class corresponding to the cycle* $\kappa^{1,1}(P)$ *is zero in* $H^*(C^1(S^1, M))$.

For $n=1$ (i.e., for 3-dimensional contact manifolds) the first part of the above theorem was proven by A. Bahri [1]. See also [1] for the definition of conjugacy and the notion of critical points at infinity.

Infinite dimensional cycles and a family of operators. In this section we define a family of operators, from which infinite dimensional cycles are derived. Since ξ is a non-zero vector field, we have a decomposition $TM = ker(\omega) \oplus \langle\xi\rangle$, and we let v be a non-zero section of $ker(\omega)$. Then we have a decomposition $ker(\omega) = \langle v\rangle \oplus W \oplus L$ for some W and some line bundle L from the assumption. For each curve $c \in C^1(S^1, M)$, we denote by $c^*(TM)$ the pullback of TM. We now define a family of operators $P = \{P_c\}$ parametrized by $C^1(S^1, M)$ as follows. For $c \in C^1(S^1, M)$, we set

$$P_c: \Gamma(S^1, c^*(TM)/\langle v\rangle)) \to \Gamma(S^1, \mathrm{Hom}(\otimes^{2n-2}W, \boldsymbol{R}) \oplus \boldsymbol{R}),$$

$$P_c(v) = \left((d\omega)^n(y, v, *, \cdots, *), \frac{d}{ds}\omega(y)\right),$$

$$y \in \Gamma(S^1, c^*TM/\langle v\rangle).$$

Although the above family of operators is not a Fredholm morphism, it is easy to get a Fredholm morphism from P by selecting appropriate subspaces of Γ. We denote again this family by P. We then have a solution cycle $\kappa^{1,1}(P)$ on the parameter space $C^1(S^1, M)$ from [3]. Actually we have cycles $\kappa^{1,1}_{p,q}(P)$, but the integers p, q depend only on the choice of subspaces in Γ for defining a Fredholm morphism. Therefore we denoted simply $\kappa^{1,1}(P)$ neglecting p, q.

References

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