33. On a Remarkable Class of Homogeneous Symplectic Manifolds

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In this note, we present some results*) on homogeneous symplectic manifolds M admitting a pair of transversal Lagrangian foliations (The class of these manifolds contains parahermitian symmetric spaces introduced and studied in [1], [2]). To such a manifold M we associate an algebraic object, called a (weak) dipolarization in a Lie algebra. We construct a natural compactification of such a manifold M arising from a semisimple graded Lie algebra. Also we give the infinitesimal classification of such manifolds corresponding to simple graded Lie algebras. The details will appear elsewhere.

1. Let M be a (connected) symplectic manifold with symplectic form ω, and let (F^+, F^-) be a pair of transversal completely integrable distributions on M. Then the triple $(M, \omega, F^{\pm})$ (or simply M) is said to be a *parakähler manifold* if each leaf of $F^{\pm}$ is a Lagrangian submanifold of M. A parakähler manifold is originally introduced by P. Libermann [4] by a different point of view (see also [1]). Let $(M, \omega, F^{\pm})$ be a parakähler manifold. By an *automorphism* of M we mean a symplectomorphism of M which leaves the distributions $F^{\pm}$ invariant. We denote by $\mathrm{Aut}\,M$ the full group of automorphisms of M, which turns out to be a finite-dimensional Lie group. If $\mathrm{Aut}\,M$ acts transitively on M, then M is called a *homogeneous* parakähler manifold. Let G be a connected Lie group and H be a closed subgroup of G. If the coset space G/H admits a parakähler structure $(\omega, F^{\pm})$ and if G acts on G/H as automorphisms, then we say that the parakähler structure $(\omega, F^{\pm})$ is *G-invariant* and that G/H is a *parakähler coset space*. A homogeneous parakähler manifold may be expressed as various parakähler coset spaces. In our situation we can consider a "parakähler algebra" which is an analogue to a Kähler algebra (Vinberg-Gindikin [5]) for a homogeneous Kähler manifold.

Definition 1. Let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{g}^{\pm}$ be two subalgebras of $\mathfrak{g}$ and ρ be an alternating 2-form on $\mathfrak{g}$. The triple $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ is called a *weak dipolarization* in $\mathfrak{g}$, if the following conditions are satisfied:

WD1) $\mathfrak{g}=\mathfrak{g}^+ + \mathfrak{g}^-$,

WD2) Put $\mathfrak{h} := \mathfrak{g}^+ \cap \mathfrak{g}^-$. Then $\rho(X, \mathfrak{g})=0$ if and only if $X \in \mathfrak{h}$,

WD3) $\rho(\mathfrak{g}^+, \mathfrak{g}^+)=\rho(\mathfrak{g}^-, \mathfrak{g}^-)=0$,

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WD4) $\rho([X, Y], Z)+\rho([Y, Z], X)+\rho([Z, X], Y)=0, \qquad X, Y, Z \in \mathfrak{g}.$

Definition 2. Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}^{\pm}$ be two subalgebras of $\mathfrak{g}$, and let f be a linear form on $\mathfrak{g}$. The triple $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ is called a *dipolarization* in $\mathfrak{g}$, if the followings are valid:

D1) $\mathfrak{g}=\mathfrak{g}^+ + \mathfrak{g}^-$,

D2) Put $\mathfrak{h} := \mathfrak{g}^+ \cap \mathfrak{g}^-$. Then $f([X, \mathfrak{g}])=0$ if and only if $X \in \mathfrak{h}$.

D3) $f([\mathfrak{g}^+, \mathfrak{g}^+])=f([\mathfrak{g}^-, \mathfrak{g}^-])=0$.

Note that a dipolarization $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ becomes a weak dipolarization just by taking df as ρ, where d denotes the coboundary operator in the sense of Lie algebra cohomology. By making use of a parakähler algebra as an intermediate, we can establish the following relationship between invariant parakähler structures on a coset space G/H and weak dipolarizations in the Lie algebra Lie G.

Theorem 1. *Let G be a connected Lie group and H be a closed subgroup of G. Let $\mathfrak{g}=\mathrm{Lie}\, G$ and $\mathfrak{h}=\mathrm{Lie}\, H$. If the coset space G/H has an invariant parakähler structure, then $\mathfrak{g}$ admits a weak dipolarization $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ satisfying*

(1) $$\mathfrak{h}=\mathfrak{g}^+ \cap \mathfrak{g}^-.$$

Conversely, suppose that there exists a weak dipolarization $\{\mathfrak{g}^+, \mathfrak{g}^-, \rho\}$ in $\mathfrak{g}$ satisfying (1) *and the following two conditions*

(2) $$(\mathrm{Ad}_{\mathfrak{g}} H)\mathfrak{g}^{\pm} \subset \mathfrak{g}^{\pm},$$

(3) $$\rho \text{ is } \mathrm{Ad}_{\mathfrak{g}} H\text{-invariant}.$$

Then G/H has an invariant parakähler structure.

Remark. In the above theorem, the conditions (2) and (3) are superfluous, provided that H is connected.

2. Let $\mathfrak{g}$ be a real semisimple Lie algebra and B be the Killing form of $\mathfrak{g}$. In this case, a weak dipolarization in $\mathfrak{g}$ is always a dipolarization, since the second cohomology group of $\mathfrak{g}$ vanishes. Later on we will be concerned with dipolarizations associated with gradations in $\mathfrak{g}$. Now let $\mathfrak{g}=\sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ be a semisimple GLA of the ν-th kind (GLA is the abbreviation of "graded Lie algebra"), and let $Z \in \mathfrak{g}$ be its characteristic element, i.e., Z is a unique element of $\mathfrak{g}$ satisfying the condition $\mathfrak{g}_k=\{X \in \mathfrak{g}: [Z, X]=kX\}$, $-\nu \leq k \leq \nu$.

Proposition 2. *Under the above situation, let $\mathfrak{g}^{\pm}=\sum_{k=0}^{\nu} \mathfrak{g}_{\pm k}$, and let f be the linear form on $\mathfrak{g}$ defined by $f(X)=B(Z, X)$, $X \in \mathfrak{g}$. Then $\{\mathfrak{g}^+, \mathfrak{g}^-, f\}$ is a dipolarization in $\mathfrak{g}$.*

From Theorem 1 and Proposition 2, we have

Theorem 3. *Let $\mathfrak{g}=\sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ be a semisimple GLA of the ν-th kind with characteristic element Z. Let G be a connected Lie group with Lie $G=\mathfrak{g}$, and $C(Z)$ be the centralizer of Z in G. Then the coset space $G/C(Z)$ has an invariant parakähler structure.*

The above coset space $G/C(Z)$ is called a *semisimple* or *simple* parakähler coset space (*of the ν-th kind*), according as G is semisimple or simple respectively.

Remark. A semisimple parakähler coset space of the ν-th kind is a parahermitian symmetric space if and only if $\nu=1$ ([1]).

Let $G/C(Z)$ be a semisimple parakähler coset space corresponding to a semisimple GLA $\mathfrak{g}=\sum_{k=-\nu}^{\nu}\mathfrak{g}_k$. Let $\mathfrak{m}^{\pm}=\sum_{k=1}^{\nu}\mathfrak{g}_{\pm k}$, and consider the parabolic subgroups $U^{\pm}=C(Z)\exp\mathfrak{m}^{\pm}$ of G and the R-spaces $M^{\pm}=G/U^{\pm}$. If G is complex semisimple, then $G/U^{\pm}$ are Kähler C-spaces in the sense of H.C. Wang. The following two theorems are generalizations of the corresponding results for parahermitian symmetric spaces [2].

Theorem 4. *A semisimple parakähler coset space $G/C(Z)$ is diffeomorphic to the cotangent bundle of the R-space G/U^+. If G is complex semisimple, then $G/C(Z)$ is holomorphically equivalent to the cotangent bundle of the Kähler C-space G/U^+.*

The group G acts on the compact product manifold $G/U^-\times G/U^+$ diagonally, that is, $g(p^-,p^+)=(gp^-,gp^+)$, where $g\in G$ and $p^{\pm}\in G/U^{\pm}$. Let $o^{\pm}$ denote the origins of $G/U^{\pm}$ respectively.

Theorem 5. *A semisimple parakähler coset space $G/C(Z)$ is equivariantly imbedded in $G/U^-\times G/U^+$ as the G-orbit through the point (o^-,o^+) under the diagonal G-action. The image of $G/C(Z)$ is open and dense in $G/U^-\times G/U^+$. In particular the compact manifold $G/U^-\times G/U^+$ is viewed as a G-equivariant compactification of $G/C(Z)$. If G is complex semisimple, then the above imbedding is holomorphic.*

3. In this paragraph we list up all the infinitesimal pairs $(\mathfrak{g},\mathfrak{g}_0)=(\mathrm{Lie}\,G, \mathrm{Lie}\,C(Z))$ corresponding to simple parakähler coset spaces $G/C(Z)$ of the second kind, which amounts to the infinitesimal classification of such spaces. This is obtained by using the results of [3] on simple GLA's.

$(\mathfrak{sl}(n,\boldsymbol{F}), \mathfrak{sl}(p,\boldsymbol{F})+\mathfrak{sl}(q,\boldsymbol{F})+\mathfrak{sl}(n-p-q,\boldsymbol{F})+\boldsymbol{F}+\boldsymbol{F})$,
$\boldsymbol{F}=\boldsymbol{R}$ or $\boldsymbol{C}$, $1\le p\le[n/2]$, $1\le q\le n-2p$, $n\ge 3$,

$(\mathfrak{sl}(n,\boldsymbol{H}), \mathfrak{sl}(p,\boldsymbol{H})+\mathfrak{sl}(q,\boldsymbol{H})+\mathfrak{sl}(n-p-q,\boldsymbol{H})+\boldsymbol{R}+\boldsymbol{R})$,
$1\le p\le[n/2]$, $1\le q\le n-2p$, $n\ge 3$,

$(\mathfrak{su}(p,q), \mathfrak{sl}(k,\boldsymbol{C})+\mathfrak{su}(p-k,q-k)+\boldsymbol{R}+i\boldsymbol{R})$, $1\le k\le p$ if $1\le p<q$, or $1\le k\le p-1$ if $3\le p=q$,

$(\mathfrak{so}(p,q), \mathfrak{sl}(k,\boldsymbol{R})+\mathfrak{so}(p-k,q-k)+\boldsymbol{R})$, $2\le k\le p$ if $2\le p<q$, or $2\le k\le p-1$ if $4\le p=q$,

$(\mathfrak{sp}(n,\boldsymbol{F}), \mathfrak{sl}(k,\boldsymbol{F})+\mathfrak{sp}(n-k,\boldsymbol{F})+\boldsymbol{F})$, $\boldsymbol{F}=\boldsymbol{R}$ or $\boldsymbol{C}$, $1\le k\le n-1$, $n\ge 3$,

$(\mathfrak{sp}(p,q), \mathfrak{sl}(k,\boldsymbol{H})+\mathfrak{sp}(p-k,q-k)+\boldsymbol{R})$, $1\le k\le p$ if $1\le p<q$, or $1\le k\le p-1$ if $2\le p=q$,

$(\mathfrak{so}^*(2n), \mathfrak{sl}(k,\boldsymbol{H})+\mathfrak{so}^*(2n-4k)+\boldsymbol{R})$, $1\le k\le(n/2)-1$ for n even≥ 6, or $1\le k\le(n-1)/2$ for n odd≥ 5,

$(\mathfrak{so}(n,\boldsymbol{C}), \mathfrak{sl}(k,\boldsymbol{C})+\mathfrak{so}(n-2k,\boldsymbol{C})+\boldsymbol{C})$, $2\le k\le[n/2]$ for n odd≥ 3, or $2\le k\le(n/2)-2$ for n even≥ 8,

$(\mathfrak{so}(n, n), \mathfrak{sl}(n-1, \boldsymbol{R})+\boldsymbol{R}+\boldsymbol{R})$,	$n\geq 4$,
$(\mathfrak{so}(2n, \boldsymbol{C}), \mathfrak{sl}(n-1, \boldsymbol{C})+\boldsymbol{C}+\boldsymbol{C})$,	$n\geq 4$,
$(E_{6(6)}, \mathfrak{sl}(5, \boldsymbol{R})+\mathfrak{sl}(2, \boldsymbol{R})+\boldsymbol{R})$	$(E_{6(6)}, \mathfrak{sl}(6, \boldsymbol{R})+\boldsymbol{R})$
$(E_{6(6)}, \mathfrak{so}(4, 4)+\boldsymbol{R})$	$(E_{6(2)}, \mathfrak{su}(3, 3)+\boldsymbol{R})$
$(E_{6(2)}, \mathfrak{so}(3, 5)+\boldsymbol{R}+i\boldsymbol{R})$	$(E_{6(-14)}, \mathfrak{su}(1, 5)+\boldsymbol{R})$
$(E_{6(-14)}, \mathfrak{so}(1, 7)+\boldsymbol{R}+i\boldsymbol{R})$	$(E_{6(-26)}, \mathfrak{so}(8)+\boldsymbol{R}+\boldsymbol{R})$
$(E_{7(7)}, \mathfrak{so}(5, 5)+\mathfrak{sl}(2, \boldsymbol{R})+\boldsymbol{R})$	$(E_{7(7)}, \mathfrak{so}(6, 6)+\boldsymbol{R})$
$(E_{7(7)}, \mathfrak{sl}(7, \boldsymbol{R})+\boldsymbol{R})$	$(E_{7(-5)}, \mathfrak{so}^*(12)+\boldsymbol{R})$
$(E_{7(-5)}, \mathfrak{so}(3, 7)+\mathfrak{su}(2)+\boldsymbol{R})$	$(E_{7(-25)}, \mathfrak{so}(2, 10)+\boldsymbol{R})$
$(E_{7(-25)}, \mathfrak{so}(1, 9)+\mathfrak{sl}(2, \boldsymbol{R})+\boldsymbol{R})$	$(E_{8(8)}, E_{7(7)}+\boldsymbol{R})$
$(E_{8(8)}, \mathfrak{so}(7, 7)+\boldsymbol{R})$	$(E_{8(-24)}, E_{7(-25)}+\boldsymbol{R})$
$(E_{8(-24)}, \mathfrak{so}(3, 11)+\boldsymbol{R})$	$(F_{4(4)}, \mathfrak{sp}(3, \boldsymbol{R})+\boldsymbol{R})$
$(F_{4(4)}, \mathfrak{so}(3, 4)+\boldsymbol{R})$	$(F_{4(-20)}, \mathfrak{so}(7)+\boldsymbol{R})$
$(G_{2(2)}, \mathfrak{sl}(2, \boldsymbol{R})+\boldsymbol{R})$	$(E_6^{\boldsymbol{C}}, \mathfrak{sl}(5, \boldsymbol{C})+\mathfrak{sl}(2, \boldsymbol{C})+\boldsymbol{C})$
$(E_6^{\boldsymbol{C}}, \mathfrak{sl}(6, \boldsymbol{C})+\boldsymbol{C})$	$(E_6^{\boldsymbol{C}}, \mathfrak{so}(8, \boldsymbol{C})+\boldsymbol{C})$
$(E_7^{\boldsymbol{C}}, \mathfrak{so}(10, \boldsymbol{C})+\mathfrak{sl}(2, \boldsymbol{C})+\boldsymbol{C})$	$(E_7^{\boldsymbol{C}}, \mathfrak{so}(12, \boldsymbol{C})+\boldsymbol{C})$
$(E_7^{\boldsymbol{C}}, \mathfrak{sl}(7, \boldsymbol{C})+\boldsymbol{C})$	$(E_8^{\boldsymbol{C}}, E_7^{\boldsymbol{C}}+\boldsymbol{C})$
$(E_8^{\boldsymbol{C}}, \mathfrak{so}(14, \boldsymbol{C})+\boldsymbol{C})$	$(F_4^{\boldsymbol{C}}, \mathfrak{sp}(3, \boldsymbol{C})+\boldsymbol{C})$
$(F_4^{\boldsymbol{C}}, \mathfrak{so}(7, \boldsymbol{C})+\boldsymbol{C})$	$(G_2^{\boldsymbol{C}}, \mathfrak{sl}(2, \boldsymbol{C})+\boldsymbol{C})$

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