25. On the Fundamental Groups of Moduli Spaces of Irreducible $SU(2)$-Connections over Closed 4-Manifolds

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§1. **Introduction and statement of result.** Let M be a connected oriented closed smooth four-manifold and $P \to M$ be a principal $SU(2)$ bundle over M with $c_2(P)=k$. Let $E=P\times_{SU(2)} C^2$ be the C^2-vector bundle associated with P by the standard representation, and $AdP=P\times_{Ad} su(2)$ be the $su(2)$ bundle associated with P by the adjoint representation. We fix integers $p\geq 2$ and $l\geq 1$. We set

$$\mathcal{A}_k := \{A+a \mid A \text{ is a smooth connection on } P.\ a\in L^p_l\Omega^1(AdP)\}$$

which is the L^p_l-completion space of the principal connections on P. Here L^p_l means the Sobolev space of sections whose derivatives of order $\leq l$ are bounded in L^p-norms, and we denote the space of AdP valued smooth m-forms on M by $\Omega^m(AdP)$. We set

$$\mathcal{G}_k := C^0(M, P\times_{Ad} SU(2)) \cap L^p_{l+1}\Omega^0(EndE)$$

which is the L^p_{l+1}-completion space of gauge group of P. We denote by $\mathcal{A}^*_k$ the subspace of irreducible connections of $\mathcal{A}_k$. We put $\mathcal{B}_k=\mathcal{A}_k/\mathcal{G}_k$ and $\mathcal{B}^*_k=\mathcal{A}^*_k/\mathcal{G}_k$. We call $\mathcal{B}^*_k$ the moduli space of irreducible $SU(2)$-connections on P. We note that $\mathcal{G}_k$ acts on $\mathcal{A}^*_k$ not freely.

In this note we study the fundamental group of $\mathcal{B}^*_k$. We shall show the following theorem.

Theorem. *Let M be a closed 4-manifold as above. Suppose that M is simply connected.*

(1) *When the intersection form of M is of odd type, then*

$$\pi_1(\mathcal{B}^*_k)=1$$

(2) *When the intersection form of M is of even type, then*

$$\pi_1(\mathcal{B}^*_k)=\begin{cases} 1 & \text{if } c_2(P)=k \text{ is odd.} \\ Z_2 & \text{if } c_2(P)=k \text{ is even.}\end{cases}$$

It is well known that S. K. Donaldson investigated the topology of 4-manifolds by using gauge theory (e.g. [2], [3]). In his works he studied the moduli space $\mathcal{M}_k$ of anti-self dual connections on P with $c_2(P)=k$. Many properties of the topology of $\mathcal{M}_k$ are got by the analysis of anti-self dual equation. But some properties are deduced from that of $\mathcal{B}^*_k$. In fact in [2] we had to show the orientability of $\mathcal{M}^*_1$. We can show it by using the fact that $\mathcal{B}^*_1$ is simply connected ([2], [4]). Further in order to get more refinement invariants of 4-manifolds we shall have to argue with moduli spaces with higher instanton number k. Therefore it is fundamental that we study the topology of $\mathcal{B}_k^*$ when we try to study the topology of 4-manifolds by dint of gauge theory.

Remark. (1) By [2] and [4] we know that $\pi_1(\mathcal{B}_1^*)=1$.

(2) Let $\mathcal{G}_k^0$ be the normal subgroup of $\mathcal{G}_k$ which fix the fibre P_{x_0} over a base point x_0 in M. Then we know the topology of the framed moduli space of connections $\tilde{\mathcal{B}}_k=\mathcal{A}_k/\mathcal{G}_k^0$ in detail. There is a weak homotopy equivalence

$$\tilde{\mathcal{B}}=Map_p(M, BSU(2))$$

where Map_p denotes the space of based maps in the homotopy class corresponding to the bundle P (see [1], [3]). Further $\tilde{\mathcal{B}}_{n,k}=\mathcal{A}_{n,k}/\mathcal{G}_{n,k}^0$ denotes the framed moduli space of connections on a principal $SU(n)$ bundle P with $c_2(P)=k$. Then $\tilde{\mathcal{B}}_{n,k}$ is simply connected for $n\geq 3$ ([2; § II.4]). These results are deduced from the topology of $\mathcal{G}_{n,k}^0$ because $\mathcal{G}_{n,k}^0$ acts freely on the contractible affine space $\mathcal{A}_{n,k}$. In fact $\pi_1(\tilde{\mathcal{B}}_{n,k})\cong\pi_0(\mathcal{G}_{n,k}^0)$. But since in our case the topology of $\mathcal{B}_k^*$ is not simply deduced from that of gauge group, we have to do more detailed argument.

(3) A. Kono proved the following result about the full gauge group $\mathcal{G}_k$ that if $\mathcal{G}_k$ is homotopy equivalent to $\mathcal{G}_{k'}$ then $k\equiv k'$ (mod 6) ([5]).

§ 2. Outline of the proof. The gauge group $\mathcal{G}_k$ has an ineffective Z_2 in its action on $\mathcal{A}_k^*$. This Z_2 is the centralizer of the holonomy subgroup of the irreducible connection on P and can be thought of as the center $\{\pm 1\}$ of $SU(2)$. These elements of the center describe elements of $\mathcal{G}_k$ because they are invariant under the adjoint action of $SU(2)$, which is used to define $\mathcal{G}_k$.

We set $\tilde{\mathcal{G}}_k=\mathcal{G}_k/Z_2$. Then we have a principal fibration

$$\tilde{\mathcal{G}}_k\longrightarrow\mathcal{A}_k^*\longrightarrow\mathcal{B}_k^*.$$

By the homotopy exact sequence of this fibration and the fact that $\mathcal{A}_k^*$ has the weak homotopy type of a point, we have

$$\pi_1(\mathcal{B}_k^*)\cong\pi_0(\tilde{\mathcal{G}}_k). \tag{1}$$

Thus we compute $\pi_0(\tilde{\mathcal{G}}_k)$.

First we compute $\pi_0(\mathcal{G}_k)$. According to [4] we have

$$\pi_0(\mathcal{G}_k)=[M, SU(2)]=[M, S^3]$$

where $[M, SU(2)]$ means the homotopy equivalence class of continuous maps from M to $SU(2)$. Moreover due to Steenrod's classification theorem (for example, see [6]) implies that

$$[M, S^3]\cong H^4(M, Z_2)/\text{Image } Sq^2$$

where $Sq^2: H^2(M, Z_2)\to H^4(M, Z)\cong Z\to Z_2$ is Steenrod's squaring operator reduced to mod 2, which is given by $Sq^2(\alpha)=\alpha\cup\alpha$ (mod 2) for $\alpha\in H^2(M, Z)$. Therefore we have

$$\pi_0(\mathcal{G}_k)\cong\begin{cases}1 & \text{if the intersection form of } M \text{ is of odd type.}\\ Z_2 & \text{if the intersection form of } M \text{ is of even type.}\end{cases} \tag{2}$$

On the other hand we have the principal fibration

$$Z_2\xrightarrow{\ j\ }\mathcal{G}_k\longrightarrow\tilde{\mathcal{G}}_k.$$

We obtain the exact sequence of pointed sets

(3) $$\longrightarrow Z_2 \xrightarrow{j_*} \pi_0(\mathcal{G}_k) \longrightarrow \pi_0(\tilde{\mathcal{G}}_k) \longrightarrow 1$$
where $Z_2 \cong \pi_0(Z_2)$. When the intersection form of M is of odd type, (1), (2) and (3) implies the assertion (1) of Theorem. When it is of even type, we have to study the map j_* in (3). We shall see the image of a non trivial element -1 of Z_2 under the map j_*.

Given any degree one map σ from M to S^4, there is a pullback $\sigma^*: [S^4, S^3] \cong Z_2 \rightarrow [M, S^3]$. Then $[M, S^3] = \text{Image } \sigma^*$. Moreover the inclusion $i: \mathcal{G}_k^0 \hookrightarrow \mathcal{G}_k$ induces an isomorphism

$$i_*: \pi_0(\mathcal{G}_k^0) \xrightarrow{\sim} \pi_0(\mathcal{G}_k)$$

by Lemma 5.10 in [4]. Thus we have the following isomorphisms

(4) $$Z_2 \cong \pi_0(\mathcal{G}_k^0) \underset{i_*}{\xrightarrow{\sim}} \pi_0(\mathcal{G}_k) \cong [M, S^3] = \text{Image } \sigma^*[S^4, S^3].$$

Now there is an open cover $M = M^+ \cup M^-$ with $M^+ \simeq B^4$ (the 4-ball), $M^+ \cap M^- \simeq S^3 \times (0, 1)$ and a clutching map $h: M^+ \cap M^- \rightarrow SU(2)$ so that the $SU(2)$-bundle P is

$$P = M^+ \times SU(2) \sqcup M^- \times SU(2) / \sim$$

where $(m^+, g) \sim (m^-, g')$ if and only if $m^+ = m^-$ and $g' = h(m^+)g$. By that $c_2(P) = k$, the map $S^3 \ni x \mapsto h(x, t) \in SU(2)$ has degree k for any $t \in (0, 1)$. Then since $\mathcal{G}_k^0$ is considered as

$$\mathcal{G}_k^0 = \{s \in \mathcal{G}_k \,|\, s | B^4 \equiv 1\}.$$

$s \in \mathcal{G}_k^0$ can be described as the pair of maps

$$s^+: M^+ \longrightarrow SU(2), \qquad s^-: M^- \longrightarrow SU(2)$$

with $s^-(x, t) = x^k s^+(x, t) x^{-k}$ on $M^+ \cap M^-$ and $s | M^+ \equiv 1$. Here we consider S^3 as the unit sphere in quaternion plane $\boldsymbol{H}$.

Let $\lambda(t) = e^{it\pi}$ $(0 \leq t \leq 1)$ be a half circle from $\lambda(0) = 1$ to $\lambda(1) = -1$ in $SU(2)$ which is also considered as the unit sphere in $\boldsymbol{H}$. We put

$$s^+ = \begin{cases} \lambda(t) & \text{on } M^+ \cap M^- = S^3 \times (0, 1) \\ 1 & \text{on } M^+ - M^- = B^4 \end{cases}$$

$$s^- = \begin{cases} x^k \lambda(t) x^{-k} & \text{on } M^+ \cap M^- = S^3 \times (0, 1) \\ -1 & \text{on } M^- - M^+. \end{cases}$$

Then this pair of maps defines an element s of $\mathcal{G}_k^0$ which is contained in the connected component of -1 in $\mathcal{G}_k$. Since $j_*(-1)$ is the connected component of -1 in $\mathcal{G}_k$, we have that $j_*(-1) = [s] \in \pi_0(\mathcal{G}_k^0) \cong \pi_0(\mathcal{G}_k)$. Under the isomorphisms in (4) we shall consider $[s]$ as an element of $\sigma^*[S^4, S^3]$. We define a degree one map σ from M to S^4 to be

$$\sigma = \begin{cases} \text{north pole} & \text{on } M^+ - M^- \\ \text{south pole} & \text{on } M^- - M^+ \\ \text{projection} & \text{on } M^+ \cap M^-. \end{cases}$$

Here the projection means the natural projection from $M^+ \cap M^- = S^3 \times [0, 1]$ to $S^3 \times [0, 1] / \sim \, = \Sigma S^3 = S^4$ which is considered as the one-suspension of S^3. We define a map u from S^4 to S^3 to be

$$u: S^4 = \Sigma S^3 = S^3 \times [0, 1] / \sim \, \ni (x, t) \longmapsto x^k e^{it\pi} x^{-k} \in S^3.$$

Then it is easy to see that $[s] \in \pi_0(\mathcal{G}_k)$ corresponds to $\sigma^*[u] = [u \circ \sigma] \in [M, S^3]$. Thus we obtain the following Lemma 1.

Lemma 1. $j_*(-1)=[s]=\sigma^*[u]$.

We note that the generator of $[S^4, S^3]\cong Z_2$ is the one-suspension ΣH of the Hopf map H from S^3 to S^2 by the suspension theorem and the fact that H generates $\pi_3(S^2)\cong Z_2$. Now we denote by H_k the k-twisted Hopf map

$$H_k: S^3 \ni x \longmapsto [x^k] \in S^2 = S^3/S^1$$

and we denote its one-suspension by ΣH_k. Then we can show the following lemmas.

Lemma 2. $[u]=[\Sigma H_k]$.

Lemma 3. $[\Sigma H_k]=k[\Sigma H]$.

To show Lemma 2 we construct a homeomorphism θ of S^3 by

$$\theta: S^3=\Sigma S^2=S^3/S^1\times[0,1]/\sim\ \ni([a],t)\longmapsto ae^{it\pi}a^{-1}\in S^3.$$

It is easy to see that θ is well defined. Then the following diagram is commutative.

$$\begin{array}{ccc} S^4=\Sigma S^3=S^3\times[0,1]/\sim & \xrightarrow{\ u\ } & S^3 \\ & \underset{\Sigma H_k}{\searrow} \circlearrowleft & \uparrow \theta\ \wr \\ & & S^3 \end{array}$$

To show Lemma 3 we define the map μ_k from S^4 to S^4 by

$$\mu_k: S^4=S^3\times[0,1]/\sim\ \ni(x,t)\longmapsto(x^k,t)\in S^4.$$

Then we can show that the degree of μ_k is k and that the following diagram is commutative.

$$\begin{array}{ccc} S^4 & \xrightarrow{\ \mu_k\ } & S^4 \\ & \underset{\Sigma H_k}{\searrow} \circlearrowleft & \downarrow \Sigma H \\ & & S^3 \end{array}$$

Thus from Lemma 1, Lemma 2 and Lemma 3 we obtain

$$j_*(-1)=[s]=\sigma^*[u]=k\sigma^*[\Sigma H]\in\pi_0(\mathcal{G}_k)\cong Z_2.$$

Hence when k is even, then j_* is 0-map. When k is odd, then j_* is surjective. So we conclude the assertion (2) of Theorem from (3).

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References

[1] M. F. Atiyah and R. Bott: The Yang-Mills equations over Riemann surfaces. Philos. Trans. Roy. Soc. London, ser. A, **308**, 523–615 (1982).

[2] S. K. Donaldson: An application of gauge theory to four dimensional topology. J. Diff. Geom., **18**, 279–315 (1983).

[3] ——: Connections, cohomology and the intersection forms of 4-manifolds. ibid., **24**, 275–341 (1986).

[4] D. S. Freed and K. K. Uhlenbeck: Instantons and Four-manifolds. M. S. R. I. Publications, Springer, New York (1984).

[5] A. Kono: The homotopy types of gauge groups. Symposium in Kinosaki, Moduli Spaces and 3, 4-manifolds, pp. 78–85 (1987) (in Japanese).

[6] E. Spanier: Algebraic Topology. McGraw Hill Inc. (1966).