2. *On the Eigenfunctions for the Sturm-Liouville Equation: Viewed as Functions of the S−L Operator*

By Takao NAMBU
Department of Mathematics, Faculty of Engineering, Kumamoto University

(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1990)

§1. **Introduction.** One of typical elliptic boundary value problems is the following Sturm-Liouville equation (or $S-L$ equation) on a finite interval $a \leq x \leq b$;

$$(1)\qquad \begin{aligned} \mathcal{L}u &= -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+q(x)u=f, \qquad a<x<b, \\ \tau u &= p\frac{du}{d\nu}+\sigma u=g, \qquad x=a,\, b, \end{aligned}$$

where the function p is positive on $a \leq x \leq b$, and $d/d\nu$ indicates the outword normal differentiation at the endpoints $x=a, b$. As is well known, the study of the equation has a long history, and very many useful results, such as the asymptotic behavior of the eigenvalues and eigenfunctions for the operators $\mathcal{L}$ and τ, are known [1, 3, 5]. While promoting a study of some control theoretic problem of one-dimensional parabolic equations, the author posed a question; are the eigenfunctions for $\mathcal{L}$ and τ continuously (in an adequate topology) dependent on the coefficients p, q, and σ? Thus, the eigenfunctions are viewed as functions of $\mathcal{L}$ and τ. Continuous dependence of the eigenvalues on p, q, and σ is well known [1] (see Lemma 1.1 below). As far as the author knows, however, no answer to the above question has been obtained. It is the purpose of the paper to derive an affirmative answer to the question. The result is fundamental, and will be useful, for example, for examining stiffness of feedback control schemes of one-dimensional parabolic systems against small changes of the parameters p, q, and σ.

Set $I=(a, b)$. Let us begin with defining an operator L acting in $L^2(I)$ as follows:

$$Lu=\mathcal{L}u, \qquad u \in \mathcal{D}(L)=\{u \in H^2(I)\,;\ \tau u=0 \quad x=a, b\}.$$

All norms hereafter will be either $L^2(I)$- or $\mathcal{L}(L^2(I))$- norm unless otherwise indicated. The operator L is clearly self adjoint. It is well known that there is a set of eigenpairs $\{\lambda_n, \varphi_n\}$ such that

(i) $\sigma(L)=\{\lambda_1, \lambda_2, \cdots\}$; $-\infty<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots\to\infty$;
(ii) $L\varphi_n=\lambda_n\varphi_n$, $n \geq 1$; and
(iii) the set $\{\varphi_n\,;\ n \geq 1\}$ forms a CONS in $L^2(I)$.

When $\mathcal{L}$ and τ are perturbed, the resultant operators will be written as

$$\tilde{\mathcal{L}}u=-\frac{d}{dx}\left(\tilde{p}(x)\frac{du}{dx}\right)+\tilde{p}(x)u,$$

and

$$\tilde{\tau}u=\tilde{p}\frac{du}{d\nu}+\tilde{\sigma}u$$

respectively. The operator corresponding to $\tilde{\mathcal{L}}$ and $\tilde{\tau}$ is denoted by $\tilde{L}$, and is given by

$$\tilde{L}u=\tilde{\mathcal{L}}u,\ u\in\mathcal{D}(\tilde{L})=\{u\in H^2(I)\,;\ \tilde{\tau}u=0,\ x=a,\ b\}.$$

The eigenpairs to $\tilde{L}$ are denoted by $\{\tilde{\lambda}_n, \tilde{\varphi}_n\}$. Continuity of λ_n relative to p, q, and σ is stated precisely as follows:

Lemma 1.1 [1]. *For each $n\geq 1$, λ_n is a continuous function of p, $q\in C(\bar{I})$, and σ. Thus, $\tilde{\lambda}_n\to\lambda_n$ as $\|\tilde{p}-p\|_{C(I)}+\|\tilde{q}-q\|_{C(I)}+|\tilde{\sigma}-\sigma|_\Gamma\to 0$, where $|\tilde{\sigma}-\sigma|_\Gamma=|\tilde{\sigma}(a)-\sigma(a)|+|\tilde{\sigma}(b)-\sigma(b)|$.*

§2. Main result. With no loss of generality, we may assume that both $\tilde{\varphi}_n$ and φ_n are real-valued functions. In order to compare $\tilde{\varphi}_n$ with φ_n, we may choose $\tilde{\varphi}_n$ for each $\tilde{p}$, $\tilde{q}$, and $\tilde{\sigma}$ such that

$$\langle\tilde{\varphi}_n, \varphi_n\rangle\geq 0, \qquad n\geq 1, \tag{2}$$

where $\langle\cdot,\cdot\rangle$ indicates the inner product in $L^2(I)$. Our main result is stated in the following theorem, where m indicates a nonnegative integer.

Theorem 2.1. *Suppose that $\tilde{\varphi}_n$ are chosen to satisfy the condition* (2). *Suppose further that p, $\tilde{p}\in C^{m+1}(\bar{I})$, and q, $\tilde{q}\in C^m(\bar{I})$. Then, we have an estimate for each $n\geq 1$*

$$\|\tilde{\varphi}_n-\varphi_n\|_{H^{m+2}(I)}\leq \text{const}\,\delta_m,$$

if δ_m is small enough, where

$$\delta_m=\|\tilde{p}-p\|_{C^{m+1}(I)}+\|\tilde{q}-q\|_{C^m(I)}+|\tilde{\sigma}-\sigma|_\Gamma.$$

Outline of the proof. Let C_n be a circle in the complex plane with center at λ_n and radius $r_n=\frac{1}{2}\min(\lambda_n-\lambda_{n-1}, \lambda_{n+1}-\lambda_n)$. It follows from Lemma 1.1 that, if δ_m is small enough, only $\tilde{\lambda}_n$ is inside C_n and all the other $\tilde{\lambda}_l$'s outside C_n. Let P_n and $\tilde{P}_n$ be projection operators defined by

$$P_n=\frac{1}{2\pi i}\int_{C_n}(\lambda-L)^{-1}d\lambda \quad\text{and}\quad \tilde{P}_n=\frac{1}{2\pi i}\int_{C_n}(\lambda-\tilde{L})^{-1}d\lambda$$

respectively. We will estimate the difference $P_n-\tilde{P}_n$, i.e., $(\lambda-L)^{-1}-(\lambda-\tilde{L})^{-1}$ on C_n. Let us introduce an auxiliary operator $\hat{L}$ as

$$\hat{L}u=\tilde{\mathcal{L}}u, \qquad u\in\mathcal{D}(\hat{L})=\mathcal{D}(L).$$

The operator $\hat{L}$ is also self adjoint, and the eigenvalues for $\hat{L}$ have a property similar to that in Lemma 1.1. It is easy to see that the resolvents for L, $\tilde{L}$, and $\hat{L}$ satisfy

$$\|(\lambda-L)^{-1}\|,\quad \|(\lambda-\tilde{L})^{-1}\|,\quad \|(\lambda-\hat{L})^{-1}\|\leq\frac{\text{const}}{r_n},\quad \lambda\in C_n.$$

Let c be a positive constant, and set $L_c=L+c$ and $\hat{L}_c=\hat{L}+c$ so that both $\sigma(L_c)$ and $\sigma(\hat{L}_c)$ entirely lie in the positive real axis. Note that $L_c^{-1}\in\mathcal{L}(H^m(I)\,;\ H^{m+2}(I)\cap\mathcal{D}(L))$. Then, we see that

$$\|1-\hat{L}_cL_c^{-1}\|_{\mathcal{L}(H^m(I))}\leq\text{const}\,\delta_m.$$

The operator $L_c\hat{L}_c^{-1}$ belongs to $\mathcal{L}(H^m(I))$. The above inequality implies that, if δ_m is small enough, the $\mathcal{L}(H^m(I))$-norm of $L_c\hat{L}_c^{-1}$ is uniformly bounded. As a result, we see that, for any $f\in H^m(I)$

$$\|(\lambda-\hat{L})^{-1}f\|_{H^{m+2}(I)}\leq \text{const}\,\|\hat{L}_c(\lambda-\hat{L})^{-1}f\|_{H^m(I)}$$
$$\leq \text{const}\,\{\|(\lambda-\hat{L})^{-1}\|_{H^m(I)}+\|f\|_{H^m(I)}\}.$$

Thus, we have the following lemma by induction:

Lemma 2.2. *If δ_m is small enough, we have an estimate*

$$\|(\lambda-\hat{L})^{-1}f\|_{H^{m+2}(I)}\leq \text{const}\,\|f\|_{H^m(I)},\quad \lambda\in C_n,\quad f\in H^m(I).$$

In Lemma 2.2, however, we have used Heinz's inequality [2] for self-adjoint $L_c^{1/2}$ and $\hat{L}_c^{1/2}$ and the moment inequality for $\hat{L}_c$ when m is an odd integer.

For a given pair of real numbers $g=\{g(a),\ g(b)\}$, consider the boundary value problem

$$(\lambda-\tilde{\mathcal{L}})u=0,\quad \tau u=g,\quad \lambda\in\rho(\hat{L}).$$

A unique solution to the problem is denoted as $u=\hat{N}(\lambda)g$. The solution is expressed, for example, as

$$\hat{N}(\lambda)g=Rg-(\lambda-\hat{L})^{-1}(\lambda-\tilde{\mathcal{L}})Rg,$$

where Rg indicates a smooth function satisfying $\tau Rg=g$ at $x=a,\ b$; for example, it is given in an elementary manner as $(Rg)(x)=(x-a)(x-b)(\alpha x-\beta)$, α and β being suitable linear combinations of g so that $\tau Rg=g$. Then,

Lemma 2.3. *The operator $\hat{N}(\lambda)$ satisfies an estimate*

$$\|\hat{N}(\lambda)g\|_{H^{m+2}(I)}\leq \text{const}\,|g|_\Gamma=\text{const}\,\{|g(a)|+|g(b)|\},\qquad \lambda\in C_n.$$

Based on the above lemmas, we will evaluate $(\lambda-L)^{-1}-(\lambda-\tilde{L})^{-1}$ for $\lambda\in C_n$. For $u\in H^m(I)$, we calculate as

$$\begin{aligned}&(\lambda-L)^{-1}u-(\lambda-\tilde{L})^{-1}u\\ &=(\lambda-L)^{-1}(L-\hat{L})(\lambda-\hat{L})^{-1}u+(\lambda-\hat{L})^{-1}u-(\lambda-\tilde{L})^{-1}u\\ &=(\lambda-L)^{-1}(1-\hat{L}_cL_c^{-1})L_c(\lambda-\hat{L})^{-1}u+\hat{N}(\lambda)\Big(\frac{p}{\tilde{p}}\tilde{\sigma}-\sigma\Big)(\lambda-\tilde{L})^{-1}u.\end{aligned}$$

Evaluation of the first term is easy. In order to evaluate the second, we remark that $\|L_c^{1/2}\tilde{L}_c^{-1/2}\|$ is uniformly bounded. This result has been obtained in [4, Lemma 2.3] under a more general situation. According to Lemma 2.3, the $H^{m+2}(I)$-norm of the above second term is upper bounded by

$$\begin{aligned}&\text{const}\,\{|\tilde{\sigma}-\sigma|_\Gamma+|\tilde{p}-p|_\Gamma\}\,\|L_c^{1/2}(\lambda-\tilde{L})^{-1}u\|\\ &\leq \text{const}\,\{|\tilde{\sigma}-\sigma|_\Gamma+|\tilde{p}-p|_\Gamma\}\,\|u\|\end{aligned}$$

via the moment inequality for $\tilde{L}_c$. Combining these results, we have obtained

$$\|(\lambda-L)^{-1}u-(\lambda-\tilde{L})^{-1}u\|_{H^{m+2}(I)}\leq \text{const}\,\delta_m\,\|u\|_{H^m(I)},\qquad \lambda\in C_n.$$

Before going back to $P_n-\tilde{P}_n$, we need one more lemma.

Lemma 2.4. *For each integer $n\geq 1$, the $H^{m+2}(I)$-norm of $\tilde{\varphi}_n$ is uniformly bounded relative to δ_m.*

The proof of Lemma 2.4 is carried out via the estimates in Lemmas 1.1 and 2.3. Note that $P_nu=\langle u,\varphi_n\rangle\varphi_n$ and $\tilde{P}_nu=\langle u,\tilde{\varphi}_n\rangle\tilde{\varphi}_n$. Then,

$$\|\langle u,\varphi_n\rangle\varphi_n-\langle u,\tilde{\varphi}_n\rangle\tilde{\varphi}_n\|_{H^{m+2}(I)}\leq \text{const}\,\delta_m\,\|u\|_{H^m(I)}$$

for $u\in H^m(I)$. Setting $u=\varphi_n$ and $\tilde{\varphi}_n$ respectively in the above inequality, we find that

$$\|\{1+\langle\tilde{\varphi}_n, \varphi_n\rangle\}(\tilde{\varphi}_n-\varphi_n)\|_{H^{m+2}(I)} \leq \text{const}\, \delta_m.$$

Owing to the assumption (2), however, this shows the desired estimate.

Q.E.D.

A possible alternative way to estimate $\varphi_n-\tilde{\varphi}_n$ will be to introduce new independent and dependent variables [1, 3]. By this transformation, the principal part of L becomes $-d^2/dx^2$ while $q(x)$ and σ take different forms. Thus, the problem will become somewhat simple. We have to assume, however, that $p(x)$ is at least of class C^2 in this case.

References

[1] R. Courant and D. Hilbert: Methods of Mathematical Physics. vol. 1, Interscience, New York (1953).
[2] T. Kato: Notes on some inequalities for linear operators. Math. Ann., **125**, 208–212 (1952).
[3] B. M. Levitan and I. S. Sargsjan: Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators. Amer. Math. Soc., Providence (1975).
[4] T. Nambu: Strong continuity of the solution to the Ljapunov equation $XL-BX=C$ relative to an elliptic operator L. Proc. Japan Acad., **65**, 70–73 (1989).
[5] E. C. Titchmarsh: Eigenfunction Expansions: Associated with Second Order Differential Equations. part 1, Oxford Univ. (Clarendon) Press, Oxford (1962).