15. *Invariants and Hodge Cycles. IV*

By Michio KUGA,*) Walter PARRY,**) and Chih-Han SAH*)

(Communicated by Shokichi IYANAGA, M. J. A., Feb. 13, 1990)

The principal goal of the present note is to provide the proof of the main theorem of [2]. The notation of [2] will be followed faithfully with only a minimum amount of recall. Let Q be a stable picture of a polyhedron $\boldsymbol{P}$ with S denoting the set of vertices of $\boldsymbol{P}$. Our target is the asymptotic behavior of

(1) $$a_r^Q=\textstyle\sum_{\vec{m}\in A(Q),|\vec{m}|=r} f(\vec{m}),$$

where

(2) $$f(\vec{x})=\prod_{\alpha\in S}\frac{2\Gamma(x^F(\alpha))}{\Gamma\left(\dfrac{x^F(\alpha)}{2}\right)\Gamma\left(\dfrac{x^F(\alpha)}{2}+2\right)}\cdot\prod_{1\le i\le k}\frac{1}{\Gamma(x_i+1)}.$$

Two functions of a positive real variable r will be said to be *asymptotically equal* if their ratio approaches 1 as $r\to\infty$. In a separate paper, we will study the generating function defined by a_r as well as the differential equations satisfied by these generating functions. The numbering of the sections present note will continue that of [2].

§ 5. **Gamma function estimates.** The purpose of this section is to gather some lemmas in preparation for the following section. We will use the following three results. The first two can be found in Artin [1]. The third is a calculus exercise.

(5.1) *Stirling's Formula,*

$$n!=n^n\cdot e^{-n}\cdot(2\pi n)^{1/2}\cdot(1+O(n^{-1}))\qquad \text{as } n\longrightarrow\infty.$$

$$\Gamma(x)=(2\pi)^{1/2}\cdot x^{x-1/2}\cdot e^{-x}\cdot(1+O(x^{-1}))\qquad \text{as } x\longrightarrow\infty.$$

(5.2) Let C denote Euler's constant. Then,

$$\frac{d}{dx}\log(\Gamma(x))=-C-\frac{1}{x}+\sum_{n\ge 1}\left\{\frac{1}{n}-\frac{1}{n+x}\right\}.$$

(5.3) Let $g\colon[\frac{1}{2},\infty)\to(0,\infty)$ be a function with $g'(t)<0$, $g''(t)>0$ for $t\ge 1/2$. Suppose further that $\int_{1/2}^{\infty}g(t)dt<\infty$. Then,

$$0<\int_{1/2}^{\infty}g(t)dt-\sum_{n\ge 1}g(n)<\{g(1/2)-g(1)\}/4.$$

From (5.1) and the functional equation of Γ, we obtain

(5.4) As $x\to\infty$, we have

$$\frac{2\Gamma(x)}{\Gamma\left(\dfrac{x}{2}\right)\cdot\Gamma\left(\dfrac{x}{2}+2\right)}=\frac{4\cdot 2^x}{(2\pi)^{1/2}\cdot x^{3/2}}\{1+O(x^{-1})\}.$$

*) Department of Mathematics, SUNY at Stony Brook, NY 11794-3651.
**) Department of Mathematics, Eastern Michigan University, Ypsilanti, MI 48197.

(5.5) For $x>0$, we have

$$\frac{d}{dx}\log\left\{\frac{2\Gamma(x)}{\Gamma\left(\frac{x}{2}\right)\cdot\Gamma\left(\frac{x}{2}+2\right)}\right\}<\log\left\{\frac{2x+1}{x+1}\right\}-\frac{1}{x+2}.$$

Proof. By using (5.2) and the functional equation of Γ, the left hand side is

$$\sum_{n\geq 1}\left\{\frac{1}{\frac{x}{2}+n}-\frac{1}{x+n}\right\}-\frac{1}{x+2}.$$

By applying (5.3) to the function

$$g(t)=\frac{1}{\frac{x}{2}+t}-\frac{1}{x+t},$$

we obtain (5.5) (with an error of $O(x^{-2})$).

(5.6) For $x\geq 0$, we have

$$\frac{2\Gamma(x)}{\Gamma\left(\frac{x}{2}\right)\cdot\Gamma\left(\frac{x}{2}+2\right)}>\frac{4\cdot 2^{x}}{(2\pi)^{1/2}\cdot(x+2)^{3/2}}.$$

Proof. Let R denote the ratio of the left hand side to the right hand side. By using (5.4), $R\to 1$ as $x\to\infty$. It is therefore enough to show that $(dR/dx)<0$ for $x>0$. We can replace R by $\log R$ in this argument. Applying (5.5), the derivative of $\log R$ is seen to be less than

$$\log\left(1-\frac{1}{2x+2}\right)+\frac{1}{2x+4}.$$

The desired assertion now follows from the simple result that $\log(1+t)\leq t$ holds for $|t|<1$.

§6. Asymptotic behavior of a_r. Proof of the main theorem. In this section, we will determine the asymptotic behavior of a_r announced in §4 of [2], see below. We begin with a definition. For every stable picture Q, set

$$b_r^Q=\left[\frac{4}{(2\pi)^{1/2}}\right]^m\left[\frac{m}{\nu r+2m}\right]^{3m/2}\frac{1}{r!}\cdot\frac{d^r}{dt^r}\{\prod_{X_i\in Q}\sinh(2^{\nu_i}t)\prod_{X_i\notin Q}\cosh(2^{\nu_i}t)\}|_{t=0}$$

and

$$b_r=\textstyle\sum_{Q\in\Pi}b_r^Q,\qquad\text{where }\nu=\max\{\nu_1,\cdots,\nu_k\}.$$

Now we show that b_r^Q is a lower bound for a_r^Q.

Theorem 6.1. *For any stable picture Q, $b_r^Q\leq a_r^Q$, so that $b_r\leq a_r$.*

Proof. By (1), (2) and (5.6), we have

$$a_r^Q\geq\left[\frac{4}{(2\pi)^{1/2}}\right]^m\sum_{\vec{m}\in A(Q),|\vec{m}|=r}\left[\frac{1}{m_1!\cdots m_k!}\prod_{\alpha\in S}\frac{2^{m^F(\alpha)}}{(m^F(\alpha)+2)^{3/2}}\right].$$

By using the geometric-arithmetic mean inequality, we have

$$\prod_{\alpha\in S}(m^F(\alpha)+2)^{1/m}\leq\frac{1}{m}\sum_{\alpha\in S}(m^F(\alpha)+2)=\frac{1}{m}\{2m+\sum_{1\leq i\leq k}\nu_i m_i\}\leq\frac{1}{m}(\nu r+2m).$$

Thus, we have

$$a_r^Q \geq \left[\frac{4}{(2\pi)^{1/2}}\right]^m \left[\frac{m}{\nu r+2m}\right]^{3m/2} \sum_{\vec{m}\in A(Q),\, |\vec{m}|=r} \prod_{1\leq i\leq k} \frac{2^{\nu_i m_i}}{m_i\,!}.$$

The theorem follows from the fact that the sum of products above is the r-th coefficient of the power series expansion of $\prod_{X_i\in Q}\sinh(2^{\nu_i}t)\cdot\prod_{X_i\notin Q}\cosh(2^{\nu_i}t)$.

Remark 6.2. In practice, the factor of b_r^Q arising from the hyperbolic sines and cosines is not difficult to evaluate. Namely, write out sinh and cosh in terms of exp, expand the product, differentiate the resulting Laurent polynomial in e^t and then set $t=0$. This shows that the dominant part of this factor is gotten by replacing both $\sinh(z)$ and $\cosh(z)$ by $e^z/2$, so we have the

Corollary 6.3. *Set $M=\sum_{1\leq i\leq k}2^{\nu_i}$. Then b_r^Q is asymptotically equal to*

$$2^{-k}\cdot\left[\frac{4}{(2\pi)^{1/2}}\right]^m\cdot\left[\frac{m}{\nu r}\right]^{3m/2}\cdot\frac{M^r}{r\,!}.$$

The above gives an asymptotic lower bound for a_r^Q. To determine the asymptotic behavior of a_r^Q better, we introduce the following notation. For $r\geq 0$, let

$$\Omega(r)=\{\vec{x}\in \boldsymbol{R}^k \mid x_1+\cdots+x_k=r \text{ and } x_i\geq 0,\ 1\leq i\leq k\}.$$

Given $\vec{\varepsilon}=(\varepsilon_1,\cdots,\varepsilon_k)\in\boldsymbol{R}^k$, let

$$\Omega_{\vec{\varepsilon}}(r)=\{\vec{x}\in\Omega(r)\mid x_i\geq\varepsilon_i r,\ 1\leq i\leq k\}.$$

For convenience, we repeat the statement of the main theorem.

Main theorem. *Let $M=\sum_{1\leq i\leq k}2^{\nu_i}$ and let $\vec{\omega}=(\omega_1,\cdots,\omega_k)$ with $\omega_i=2^{\nu_i}\cdot M^{-1}$, $1\leq i\leq k$. Let Q be any stable picture. Then a_r^Q is asymptotically equal to*

$$2^{-k}\cdot 4^m\cdot(2\pi)^{-m/2}\cdot\prod_{\alpha\in S}\omega^F(\alpha)^{-3/2}\cdot\frac{M^r}{r^{3m/2}\cdot r\,!}.$$

Proof. First choose $\vec{\varepsilon}\in\boldsymbol{R}^k$ so that $0\leq\varepsilon_i<\omega_i$, $1\leq i\leq k$. Then

(6.4) $$a_r^Q=\textstyle\sum_{\vec{m}\in A(Q)\cap\{\Omega(r)-\Omega_{\vec{\varepsilon}}(r)\}}f(\vec{m})+\sum_{\vec{m}\in A(Q)\cap\Omega_{\vec{\varepsilon}}(r)}f(\vec{m}).$$

We will prove the following assertion.

(6.5) There is a positive real number $\tilde{M}<M$ for which the first sum in (6.4) is asymptotically less than $\tilde{M}^r/r\,!$

To prove (6.5), we first define $\vec{\varepsilon}_i$ to be the vector in $\boldsymbol{R}^k$ whose i-th component is ε_i and whose other components are 0. Thus, $\vec{\varepsilon}=\sum_i\vec{\varepsilon}_i$, and we have

(6.6) $$\text{first sum in (6.4)}\leq\textstyle\sum_{1\leq i\leq k}\sum_{\vec{m}\in A(Q)\cap\{\Omega(r)-\Omega_{\vec{\varepsilon}_i}(r)\}}f(\vec{m}).$$

Combining (5.4) with the fact that $\Gamma(x)$ has a simple pole with residue 1 at $x=0$, we see that $2\Gamma(x)\cdot 2^{-x}\cdot\{\Gamma(x/2)\cdot\Gamma((x/2)+2)\}^{-1}$ is bounded. This and (6.6) show that we are reduced to proving (6.5) for sums of the form

$$\sum_{\vec{m}\in \boldsymbol{Z}^k\cap\{\Omega(r)-\Omega_{\vec{\varepsilon}_i}(r)\}}\prod_{1\leq j\leq k}\frac{2^{\nu_j m_j}}{m_j\,!}.$$

With an "obvious" interpretation, the above sum has the form

$$\frac{1}{r\,!}\sum_{0\leq j\leq\varepsilon_i r}\begin{bmatrix} r\\ j\end{bmatrix}a^{r-j}b^j,\quad a=M-2^{\nu_i},\quad b=2^{\nu_i}.$$

For small values of ε, the terms in the above sum of the form $\begin{bmatrix} r\\ \varepsilon r\end{bmatrix}a^{(1-\varepsilon)r}b^{\varepsilon r}$ are small and pose no difficulty. For the larger values of ε (close to ε_i), a straightforward application of Stirling's formula shows that for $0 \ll \varepsilon \leq \varepsilon_i$ (≤ 1) and $r \to \infty$, such terms are $O((a/(1-\varepsilon))^{(1-\varepsilon)r} \cdot (b/\varepsilon)^{\varepsilon r})$. This brings us to the following lemma whose proof is an elementary exercise.

Lemma. *Let a, b be positive real numbers. Then, as ε increases from 0 to $(b/(a+b))$, the expression $(a/(1-\varepsilon))^{1-\varepsilon} \cdot (b/\varepsilon)^{\varepsilon}$ increases from a to $a+b$.*

In the situation at hand, $a+b=M$ and $(b/(a+b))=(2^{\nu_i}/M)=\omega_i>\varepsilon_i \geq \varepsilon$. The assertion (6.5) now follows easily.

With (6.5) at hand, Corollary 6.3 above shows that a_r^Q is asymptotically equal to

$$\sum f(\vec{m})=\sum \prod_{\alpha \in S} \frac{2\Gamma(m^F(\alpha)) \cdot 2^{-m^F(\alpha)}}{\Gamma\left(\dfrac{m^F(\alpha)}{2}\right) \cdot \Gamma\left(\dfrac{m^F(\alpha)}{2}+2\right)} \prod_{1 \leq i \leq k} \frac{2^{\nu_i m_i}}{m_i!}. \tag{6.7}$$

Here both sums extend over $\vec{m} \in A(Q) \cap \Omega_{\hat{\varepsilon}}(r)$. Arguing as in Remark 6.2, the expression

$$\sum_{\vec{m} \in A(Q) \cap \Omega(r)} \prod_{1 \leq i \leq k} \frac{2^{\nu_i m_i}}{m_i!}$$

is asymptotically equal to $2^{-k} \cdot M^r/r!$. The argument presented before the above lemma shows that we have the same asymptotic value $2^{-k} \cdot M^r/r!$ when we replace $\Omega(r)$ by $\Omega_{\hat{\varepsilon}}(r)$ in the preceding expression. Finally, we consider (6.7). By applying the estimate of (5.4), letting $\hat{\varepsilon}$ converge to $\vec{\omega}$ and noting that the sets $\Omega_{\hat{\varepsilon}}(r)$ converge to the point $r\vec{\omega}$, the statement of the main theorem then follows.

Corollary 6.8. *The asymptotic value of a_r^Q is independent of the stable picture Q.*

Corollary 6.9. *We have that a_r is asymptotically equal to*

$$\frac{n(P)}{2^k}\left[\frac{4}{(2\pi)^{1/2}}\right]^m \cdot \prod_{\alpha \in S} \omega^F(\alpha)^{-3/2} \cdot \frac{M^r}{r^{3m/2} \cdot r!}.$$

Corollary 6.10. *If $\nu_i=\nu$ and $f_i=f$ hold for $1 \leq i \leq k$, then a_r is asymptotically equal to b_r.*

Proof. A comparison of Corollaries 6.3 and 6.9 shows that we must prove that

$$\left[\frac{m}{\nu}\right]^m=\prod_{\alpha \in S} \omega^F(\alpha)^{-1}=\prod_{\alpha \in S} \frac{k}{f}=\left[\frac{k}{f}\right]^m,$$

and so we must prove that $m/\nu=k/f$. This follows easily from the fact that there are k faces each with ν vertices and each vertex appears in f faces.

Added in Proof. Professor Michio Kuga deceased in February 15, 1990 at the age of 62 years.

References

[1] E. Artin: The Gamma Function. Holt, Rinehart and Winston (1964).
[2] M. Kuga, W. Parry, and C.-H. Sah: Invariants and Hodge cycles. III. Proc. Japan Acad., **66**A, 22–25 (1990).