78. On Certain Real Quadratic Fields with Class Numbers 3 and 5

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Let d be a square-free integer of the form $d=r^2+4$ or r^2+1 $(r\in \boldsymbol{N})$. Recently H.K. Kin, M.-G. Leu and T. Ono proved in [4] that there exist 11 real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ with class number one with at most one exception of d. In [5] Leu proved further that there exist 16 real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ of class number 2 with at most one exception of d.

In this paper, we shall consider the fields of the same kind with class numbers 3 and 5. Our main theorem is obtained by using almost the same methods as in [4], [5] and by the help of a computer.

We denote the class number of the quadratic field $\boldsymbol{Q}(\sqrt{d})$ by $h(d)$, which we shall abbreviate to h, if there is no fear of confusion. By the genus theory of quadratic fields, we have the following lemma.

Lemma 1. *If d is square-free and $d=r^2+4$ or r^2+1 and $h(d)$ is odd, then d is a prime.*

In the following, we restrict ourselves to the case $h(d)=3$ or 5. Therefore d is a prime and is denoted by p.

Then p should be expressed in the form $p=m^2+4$ (m: odd) or $p=4m^2+1$ $(m\in\boldsymbol{N})$. We note here that $p\equiv 1(4)$ and the fundamental unit $\varepsilon_p=(t+u\sqrt{p})/2$ of $\boldsymbol{Q}(\sqrt{p})$ is given with $t=m$, $u=1$, in the case $p=m^2+4$, and given with $t=4m$, $u=2$ in the case $p=4m^2+1$.

Let χ_p be the Kronecker character belonging to $\boldsymbol{Q}(\sqrt{p})$ and $L(s,\chi_p)$ be the corresponding L-series. Then by Theorem 2 of [6], for any $x\geqq 11.2$ and $p\geqq e^x$, we have

$$L(1,\chi_p)>\frac{0.655}{x}p^{-1/x}$$

with one possible exception of p. From class number formula, we obtain

$$\begin{aligned} h(p)&=\frac{\sqrt{p}}{2\log\varepsilon_p}L(1,\chi_p)>\frac{0.655}{x}\;\frac{\sqrt{p}\,p^{-1/x}}{2\log(u\sqrt{p})}\\ &=\frac{0.655}{x}\;\frac{p^{(x-2)/2x}}{2\log u+\log p}>\frac{0.655\,e^{(x-2)/2}}{x(x+3)}. \end{aligned}$$

Since $f(x)=\dfrac{e^{(x-2)/2}}{x(x+3)}$ is a monotone increasing function for $x\geqq 11.2$, we have

$$h(p)>0.655f(17)=3.48\cdots>3\ (x\geqq 17),$$
$$h(p)>0.655f(18)=5.16\cdots>5\ (x\geqq 18).$$

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Theorem 1. *There exists at most one* $p \geqq e^{18}$ *with* $h(p)=1$, 3 *or* 5.

From Lemma 2 of [7], we have the following lemma.

Lemma 2. *If q is the least prime such that*

$$\left(\frac{p}{q}\right)=1, \quad \textit{then } h(p) \geqq \frac{\log m}{\log q} \quad \textit{holds.}$$

For the case $h(p)=3$, the following tables of all the primes $p=m^2+4$ and $4m^2+1$ and the least prime q and the class number h such that $p \leqq e^{17}$ and $m<q^3$ are obtained by the help of a computer. From these tables we have $h(p)=3$ if and only if $p=229$, 257, 733, 1229, 1373, 2213, 2917, 4493, 5333, 5477, 8837, 9413 with one possible exception.

For the case $h(p)=5$, there exist 651 primes p such that $p \leqq e^{18}$ and $m<q^5$. For $p=m^2+4$, there exist 413 primes $\{p=5, \cdots, 65496653\}$ such that $p \leqq e^{18}$ and $m<q^5$. For $p=4m^2+1$, there exist 238 primes $\{p=5, \cdots, 64224197\}$ such that $p \leqq e^{18}$ and $m<q^5$. In the same way as for $h(p)=3$, we have $h(p)=5$ if and only if $p=401$, 1093, 3253, 4357, 7229, 10613, 12101, 13229, 18773, 21317, 27893, 37253, 42437 with one possible exception.

Finally we obtain the following Theorem 2, which contains the results of [4].

Theorem 2. *There exist only* 36 *primes p of the form* m^2+4 *or* $4m^2+1$ *such that* $h(p)=1$, 3 *or* 5 *with one possible exception.*

(1) $h(p)=1$ $p=5$, 13, 29, 53, 173, 293 $(p=m^2+4)$,
$p=5$, 17, 37, 101, 197, 677 $(p=4m^2+1)$ (Theorem 2 of [4]).

(2) $h(p)=3$ $p=229$, 733, 1229, 1373, 2213, 4493, 5333, 9413 $(p=m^2+4)$,
$p=257$, 2917, 5477, 8837 $(p=4m^2+1)$.

(3) $h(p)=5$ $p=1093$, 3253, 7229, 10613, 13229, 18773, 27893, 37253 $(p=m^2+4)$,
$p=401$, 4357, 12101, 21317, 42437 $(p=4m^2+1)$.

Remark. Assuming the generalized Riemann Hypothesis, it is known that the conclusion of Tatuzawa's theorem is true without any exception. So if we assume this, Theorem 2 holds without any exception.

Table I ($p=m^2+4$)

p	m	q	h	p	m	q	h	p	m	q	h
5	1	5	1	85853	293	11	11	1555013	1247	11	33
13	3	3	1	94253	307	11	9	1723973	1313	11	37
29	5	5	1	97973	313	7	13	2122853	1457	29	29
53	7	7	1	100493	317	7	13	2319533	1523	17	33
173	13	13	1	120413	347	17	11	2380853	1543	19	29
229	15	3	3	162413	403	11	15	2679773	1637	29	31
293	17	17	1	165653	407	11	13	3108173	1763	13	45
733	27	3	3	214373	463	11	17	4355573	2087	13	45
1229	35	5	3	237173	487	13	11	4592453	2143	19	45
1373	37	7	3	253013	503	17	15	4721933	2173	17	45
2213	47	7	3	284093	533	13	17	4826813	2197	13	45
4229	65	5	7	299213	547	17	15	4959533	2227	17	41
4493	67	11	3	332933	577	13	15	5793653	2407	17	49
5333	73	11	3	375773	613	29	13	7219973	2687	19	53
7229	85	5	5	494213	703	11	25	8082653	2843	23	57
9029	95	5	7	508373	713	19	17	8334773	2887	17	65
9413	97	13	3	552053	743	29	15	10004573	3163	19	55
10613	103	7	5	588293	767	13	23	10804373	3287	19	63
13229	115	5	5	619373	787	13	25	11950853	3457	19	63
15629	125	5	9	677333	823	17	21	14417213	3797	23	61
18773	137	13	5	683933	827	13	19	14922773	3863	37	65
26573	163	7	9	727613	853	13	23	15816533	3977	23	69
27893	167	17	5	779693	883	19	21	17081693	4133	17	67
37253	193	23	5	908213	953	11	27	20097293	4483	19	79
41213	203	7	7	935093	967	11	25	20457533	4523	19	79
47093	217	7	9	966293	983	11	33	22534013	4747	23	85
54293	233	7	9	994013	997	11	25	22629053	4757	17	93
64013	253	11	9	1247693	1117	19	27	4951^2+4			
76733	277	19	7	1261133	1123	11	27	$=24157229>e^{17}$			
82373	287	7	13	1385333	1177	11	31				

Table II ($p=4m^2+1$)

p	m	q	h	p	m	q	h	p	m	q	h
5	1	5	1	417317	323	13	15	5541317	1177	11	51
17	2	2	1	454277	337	19	15	6728837	1297	11	75
37	3	3	1	470597	343	7	21	7540517	1373	13	63
101	5	5	1	1110917	527	11	25	9156677	1513	13	61
197	7	7	1	1136357	533	13	27	9647237	1553	17	57
257	8	2	3	1196837	547	11	25	9821957	1567	29	49
677	13	13	1	1378277	587	13	35	10666757	1633	13	71
2917	27	3	3	1705637	653	19	27	10719077	1637	17	45
5477	37	13	3	1833317	677	19	25	10982597	1657	17	61
8837	47	11	3	2515397	793	11	55	11115557	1667	19	59
12101	55	5	5	2735717	827	13	37	11874917	1723	17	53
16901	65	5	7	3118757	883	11	37	13586597	1843	19	61
17957	67	7	7	3147077	887	31	31	14092517	1877	13	69
21317	73	7	5	3587237	947	11	49	15570917	1973	23	73
28901	85	5	9	4024037	1003	13	51	15952037	1997	13	83
42437	103	23	5	4104677	1013	11	33	16273157	2017	13	77
52901	115	5	15	4218917	1027	13	45	17255717	2077	23	71
62501	125	5	9	4301477	1037	11	45	18438437	2147	13	113
98597	157	7	11	4519877	1063	17	47	19061957	2183	19	77
106277	163	31	7	4726277	1087	13	37	19483397	2207	37	71
148997	193	7	15	5134757	1133	11	67	22905797	2393	19	63
164837	203	7	13	5262437	1147	11	55				
217157	233	17	13	5354597	1157	13	43	$4\times 2477^2+1$			
401957	317	13	17	5410277	1163	11	45	$=24542117>e^{17}$			

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