61. Double Coverings of P^2

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1. Introduction. Let k be an algebraically closed field of characteristic zero (sometimes let $k=\boldsymbol{C}$). We denote by K a purely transcendental extension of k, and by L a quadratic extension of K. If $\dim_k K=1$ or equivalently if $K=k(x)$, then L can be expressed as $K(y)$, where the element y satisfies the equation $y^2=(x-a_1)\cdots(x-a_{2n+1})$ with distinct $a_i \in k$ $(i=1, \cdots, 2n+1)$. The model of L is a rational, an elliptic or a hyperelliptic curve. In this note we consider the similar subject in the case when $\dim_k K=2$. Suppose $K=k(x, y)$ and let S be a nonsingular model of L. Then we will study the structure of S from the birational viewpoint. Details will appear elsewhere. The author would like to thank Prof. F. Sakai for valuable informations. We use the following notations:

$$p_g=p_g(S)=\dim H^2(S, \mathcal{O}), \qquad q=q(S)=\dim H^1(S, \mathcal{O}),$$
$$c_i=c_i(S): \text{the } i\text{-th Chern class of } S \quad (i=1, 2).$$

2. Theorems. First we obtain the following basic theorem by making use of projective and standard Cremona transformations of $\boldsymbol{P}^2$.

Theorem 1. *We can find an element $z \in L$ and a polynomial $f(x, y) \in k[x, y]$ satisfying the following conditions:*

(1) *$L=K(z)$, where $z^2=f(x, y)$ and f is reduced.*

(2) *The degree of f is even and the curve $f=0$ in $\boldsymbol{P}^2$ has at most ordinary singularities.*

Let $d=2e$ be the degree of f. We introduce new variables z_i having the gradations $\deg z_i=1$ $(i=0, 1, 2)$ and $\deg z_3=e$. Then, putting $x=z_1/z_0$, $y=z_2/z_0$, $z=z_3/(z_0)^e$ and $F(z_0, z_1, z_2)=f(z_1/z_0, z_2/z_0)z_0^d$, we get a surface F defined by the equation $z_3^2=F(z_0, z_1, z_2)$ in the weighted projective space $\boldsymbol{P}(1, 1, 1, e)$. Let C be the plane curve defined by the equation $F(z_0, z_1, z_2)=0$. Then we have a double covering $\pi: F \to \boldsymbol{P}^2$ branched along the curve C. When C is not smooth, let P_i $(i=1, \cdots, r)$ be the singular point of C and let m_i be the multiplicity of C at P_i. Let $\sigma: X \to \boldsymbol{P}^2$ be a composition of the blow-ups with the centers $\{P_i \mid i=1, \cdots, r\}$. Then let F' be the normalization of $F \times_{\boldsymbol{P}^2} X$. In case m_i is odd, let $\{P_{ij} \mid j=1, \cdots, m_i\}$ be the points $C' \cap E_i$, where C' denotes the proper transform of C by σ^{-1} and $E_i=\sigma^{-1}(P_i)$. Let $\mu: Y \to X$ be a composition of the blow-ups with the centers $\{P_{ij} \mid m_i \text{ is odd}\}$. Then, letting S be the normalization of $F' \times_X Y$, we obtain a double covering $\tilde{\pi}: S \to Y$. Here we note that S is the minimal resolution of F and a nonsingular model of L. Put $n_i=[m_i/2]$, where [ ] denotes the Gauss' symbol. Then the geometric invariants of S are given as follows, where $\sum$ stands for $\sum_{i=1}^{r}$.

Lemma 2. (1) *The canonical divisor of S is linearly equivalent to $\tilde{\pi}^*\cdot\mu^*(\sigma^*((e-3)l)-\sum(n_i-1)E_i)$, where l is a line in $\boldsymbol{P}^2$. Hence*

$$p_g=\dim H^0(X,\ \sigma^*((e-3)l)-\sum(n_i-1)E_i).$$

(2) $q=\dim H^1(X,\ -D)$, *where* $D=\sigma^*(el)-\sum n_iE_i$.

(3) $2(p_g-q)=(e-1)(e-2)-\sum n_i(n_i-1)$.

(4) $c_1^2=2\{(e-3)^2-\sum(n_i-1)^2\}$.

(5) $c_2=2\{2e^2-3e+3-\sum(2n_i^2-n_i-1)\}$.

From the existence of the double covering $\tilde{\pi}: S\rightarrow Y$ we infer the following former assertion. The latter one is obtained from the above lemma.

Theorem 3. *Abelian and hyperelliptic surfaces cannot be birationally equivalent to double coverings of $\boldsymbol{P}^2$. Except those every class in the Enriques-Kodaira classification can appear as a model of L.*

Theorem 4. *If $e\geq 4+\sum_{i=1}^{r}(n_i-1)$, then S is a minimal surface of general type. Moreover the inequality $c_1^2/c_2<1/2$ holds true for S.*

By the vanishing theorem of Mumford [3] we obtain the following

Theorem 5. *If $e\geq 1+\sum_{i=1}^{r}n_i$, then $q=0$.*

For a fixed number m, let $\mathcal{S}_m$ be the set

$$\left\{S\ \middle|\ \begin{array}{l}S \text{ is a surface obtained as above for some } L \text{ and}\\ \text{the branch-locus } C \text{ satisfies } \sum_{i=1}^{r} m_i<m.\end{array}\right\}$$

Note that, for some L, $\mathcal{S}_m$ may contain more than one surfaces which are models of L. But, by Theorem 4, if e is large enough, then S is determined uniquely by L. By Theorems 4 and 5, we obtain the following

Theorem 6. *The surface $S\in\mathcal{S}_m$ is a minimal surface of general type with $q=0$ for every sufficiently large d and the equality $\lim_{d\to\infty}c_1^2/c_2=1/2$ holds true for the elements of $\mathcal{S}_m$.*

Remark 7. (1) In the case of special triple coverings of $\boldsymbol{P}^2$ treated in [4], we have that $\lim_{d\to\infty}c_1^2/c_2=2/3$.

(2) For a fixed number n, let $\mathcal{T}$ be the set consisting of surfaces $\{S\}$, where S is an n-cyclic covering of $\boldsymbol{P}^2$ branched along a nonsingular curve of degree dn. Then S is a minimal surface of general type if $(n-1)d>3$. We infer readily that $\lim_{d\to\infty}c_1^2/c_2=(n-1)/n$ for the elements of $\mathcal{T}$.

3. Examples. For each d let us describe the structure of S.

(1) $d=2$: S is a rational surface.

(2) $d=4$, (2–1): If $C=\bigcup_{i=1}^{4}l_i$ and $\bigcap_{i=1}^{4}l_i$ is one point, where l_i is a line, then S is a ruled surface with $q=1$. (2–2): Otherwise S is a rational surface.

(3) $d=6$, (3–1): If $m_i\leq 3$ for all i, then S is a $K3$ surface. (3–2): Otherwise S is a ruled surface with $2q+2=\sum n_i(n_i-1)$.

(4) $d\geq 8$, (4–1): If $r=0$, then S is a minimal surface of general type. (4–2): If $r=1$ and $e\geq n=n_1\geq e-1$, then S is a ruled surface with $2q=n(n-1)-(e-1)(e-2)$. (4–3): If $r=1$ and $n=e-2$, then S is a

minimal properly elliptic surface with $p_g=e-2$ and $q=0$. (See, for the definition, [1, p. 189].) (4–4): If $r=1$ and $n\leq e-3$, then S is a minimal surface of general type. $\cdots\cdots$

Here we present more concrete examples.

Example 8. Let $f_1=y+x^a$ and $f_2=x+y^a$. Put $f_j=\alpha_j f_1+\beta_j f_2$ $(j=3, \cdots, m)$, where we assume that $\alpha_j\beta_j\neq 0$ and $\alpha_i\beta_j\neq\alpha_j\beta_i$ if $i\neq j$. Let $f=f_1\cdots f_m$, then $d=am$, $r=a^2$ and $m_i=m$.

(I) In case $m=2h$, the classification is as follows:

(i) If $a=2$, then S is a ruled surface with $q=h-1$.

(ii) If $a=3$, then S is a minimal properly elliptic surface.

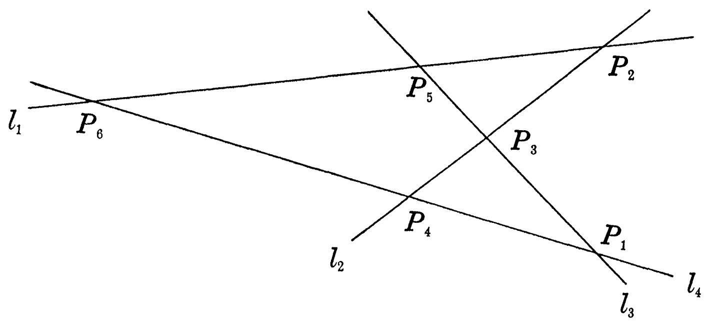
(iii) If $a\geq 4$, then S is a minimal surface of general type. When h [resp. a] is fixed, $\lim_{a\to\infty} c_1^2/c_2=(2h-1)/(h+1)$ [resp. $\lim_{h\to\infty} c_1^2/c_2=2$].

(II) In case $m=2h+1$ and $a=2b$, then $q=0$ and the classification is as follows:

(i) If $b=1$, then S is a minimal properly elliptic surface.

(ii) If $b\geq 2$, then S is a minimal surface of general type. When h [resp. b] is fixed, $\lim_{b\to\infty} c_1^2/c_2=(4h-1)/(4h+2)$ [resp. $\lim_{h\to\infty} c_1^2/c_2=(2b-2)/(2b-1)$].

Example 9. Let $\{l_i\}$ be four lines whose arrangement is described in the figure below. We can find an irreducible plane curve C_0 satisfying the following conditions: The degree of C_0 is 8 and the reducible curve $C=C_0\cup l_1\cup\cdots\cup l_4$ has only ordinary singular points P_i $(i=1, \cdots, 6)$ with $m_1=m_2=6$ and $m_3=\cdots m_6=4$. Let S be the double covering obtained from C by the procedure stated in section 2. From Lemma 2 we see that $c_1^2=-6$ and $p_g=q$, but we can find 6 exceptional curves on S. Contracting these curves, we get the minimal surface on which the bicanonical divisor is linearly equivalent to 0. It turns out that it is an Enriques surface.



4. Application. Finally we mention an application to the theory of plane curves. For given numerical characters satisfying the genus formula, a cuspidal rational curve with the characters does not necessarily exist (see, [2] and [4]). But in case the singularity is ordinary, a similar fact does not seem to be known. Let C be an irreducible rational curve of degree $2e$. Suppose that C has ordinary singular points with the same multiplicity $2n$. Then we have $(2e-1)(2e-2)=2n(2n-1)r$. Note that in case $n=1$ (i.e., the singularity is a node), such curves exist for any e (see, [5, Theorem 4 in p. 216]). But in the case $n \geq 3$ such curves do not always exist. In fact, suppose that C defines a surface of general type S and let S_0 be the minimal model. Then we have that $(c_1^2+\delta)/(c_2-\delta) \leq 3$ by the Miyaoka-Yau's inequality, where $\delta=c_1^2(S_0)-c_1^2(S)$. Hence we infer from Lemma 2 the following

Theorem 10. *In case $e \geq 3n+1$, then S is a surface of general type. Furthermore the following inequality holds true.*

$$(3n-8)e^2-(9n^2-12)e+5n^2-n-4 \leq 0.$$

In particular we have that

(i) *if $n=3$, then $e \leq 68$,*

(ii) *if $n \geq 34$, then $e \leq 3n+8$.*

Question 11. We do not know whether the inequality above is the best possible one. For example we do not know whether a curve with $23 \leq e \leq 68$ and $n=3$ exists. If such a curve exists, then for the minimal model of S the inequality $c_1^2/c_2 \geq 2$ holds true (see, [1, p. 229]).

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