7. Invariants and Hodge Cycles. III

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The present work is a continuation of [1] and [2]. Consider a GTAS (Group Theoretic Abelian Scheme) $\pi: A \to V = \Gamma \backslash \mathcal{D}$. The space
$$H^{(p,p)}(A_\lambda) \cap H^r(A_\lambda, \boldsymbol{Q}), \qquad (r = 2p)$$
of Hodge cycles in a generic fibre A_λ is controlled by an invariant theory of G, [1], [2], where G is the $\boldsymbol{Q}$-semisimple algebraic group attached to V. In fact, when A is rigid, the space $H^{(p,p)}(A_\lambda) \cap H^r(A_\lambda, \boldsymbol{Q})$ coincides with the space $H^r(A, \boldsymbol{Q})^G = \Lambda^r(F)^G$ of G-invariant elements in $H^r(A_\lambda, \boldsymbol{Q})$. Here $F = H^1(A_\lambda, \boldsymbol{Q})$. However, this invariant theory is quite different from the classical invariant theory. First, it deals with the exterior product $\Lambda^r(F)$ of the basic representation space F rather than the symmetric product $S^r(F)$ that appears in classical theory. Second, the basic representation (ρ, F) is a very special kind of representation called "rigid polymer type in a chemistry $(\mathfrak{G}, S, S_0)$" that is related to a combinatorics of a finite group $\mathfrak{G}$, [1], [2]. As a result, our invariant theory becomes quite different from the classical one and even the determination of $\dim \Lambda^r(F)^G$, the dimension of the space of invariants, becomes difficult in general, [1], [2]. However, the asymptotic behavior of $\dim \Lambda^r(\mu F)^G$ (as $\mu \to \infty$) can be studied. This will be the goal of the present work. Before going further, we would like to thank Dr. David Weeks, Mrs. Oscar Goldman and Professor Shokichi Iyanaga, M.J.A., for their encouragements.

Let a rigid GTAS $\pi: A \to V$ be of quaternion type so that it corresponds to a polyhedron (polymer) P with a generic fibre A_λ. Then the fibre product (with μ factors) $\pi^{(\mu)}: A \times_V \cdots \times_V A \to V$ corresponds to the polyhedron μP and has a generic fibre $\mu A_\lambda = A_\lambda + \cdots + A_\lambda$ (μ terms). If the basic representation space of the GTAS A is $F = H^1(A_\lambda, \boldsymbol{Q})$, then the representation space of $A \times_V \cdots \times_V A$ is $\mu F = H^1(\mu A_\lambda, \boldsymbol{Q}) = F \oplus \cdots \oplus F$. Let $F = X_1 + \cdots + X_k$ be the decomposition of F into irreducible pieces. In general, $X_i = X_j$ is permitted for $i \neq j$. However, in the present work, we will be concerned with the case where $X_i \neq X_j$ holds for $i \neq j$. We note that the more general case of $F = \mu_0 F_0$ with $\mu_0 > 1$ can be subsumed under the substitution of μ by $\mu \mu_0$ in our calculation. We recall that $[\Lambda^r(\mu X_1 + \cdots + \mu X_k)] = [\Lambda^r(\mu F)] =_{df} \dim \Lambda^r(\mu F)^G$ $\in \boldsymbol{Z}$. This function is only additive on the representation ring of G. We have
$$\Lambda^r(\mu X_1 + \cdots + \mu X_k) \cong \bigoplus_{\sum a_{i,j} = r} \bigotimes_{i,j} \Lambda^{a_{i,j}}(X_i), \quad \text{where } 1 \leq i \leq k \text{ and } 1 \leq j \leq \mu.$$
In the preceding direct sum, those summands with $a_{i,j} \leq 1$ for all i, j are said to be of the *first kind* while those with at least one $a_{i,j} > 1$ are said to be of the *second kind*. Thus,

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$$\dim \varLambda^r(\mu F)^G = \sum [\text{first kind term}] + \sum [\text{second kind term}].$$

The equality $H^{(p,p)}(A_\lambda) \cap H^r(A_\lambda, \boldsymbol{Q}) = \varLambda^r(F)^G$ is valid if the polyhedron (polymer) P belongs to some rigid GTAS and in this case $\varLambda^r(F)^G = 0$ holds for odd r. If there is no GTAS associated to P, then the equality becomes meaningless (even though a polymer representation for P still exists) and $\varLambda^r(F)^G$ might be nonzero. Later, we will discuss the general case.

In the present work, we will refer to some calculations made in [2]. There we used homology. Here we use cohomology. They are isomorphic by using the self duality of F. We will review the notation used in our earlier works. Before we begin, we note that a large part of the present work makes sense without the background involving GTAS. As a result, we often work in the general setting and there will be abuses of notation. The reader should refer to the earlier works [1], [2] for more details. Specifically, some of the symbols have multiple interpretations. The precise meaning is usually clear from context. Every so often, reminders will be inserted to clarify the usage.

Let P be a polyhedron (or polymer). This means that P is viewed both as a formal finite sum $X_1 + \cdots + X_k$ as well as a triple (S, F, V). We recall that S is the finite set of vertices and F is the finite set of "top dimensional" faces of P. Here $V: F \to 2^S$ is the vertex map, and $F = \{X_1, \cdots, X_k\}$, so that $k = |F|$. We note that $X_i = X_j$ is permissible for $i \neq j$ so that the faces of P may have multiplicities greater than 1. However, we can artificially introduce distinct symbols so that $X_i \neq X_j$ holds for $i \neq j$. As a result, X_i may be replaced by i when there is little chance of confusion. By an abuse of notation, the "open star map" will be denoted by $V^{-1}: S \to 2^F$. Thus, for the face i, $V(i)$ can be identified with the subset $X_i = V(i)$ of S consisting of the vertices of $V(i)$; for the vertex α, $V^{-1}(\alpha)$ is then the set of all faces X_j with $\alpha \in X_j$. We set $m = |S|$, $\nu_i = |X_i| = |V(X_i)|$, and $f(\alpha) = |V^{-1}(\alpha)|$. The $k \times m$ incidence matrix $A = (a(\alpha, i))$ is defined by $a(\alpha, i) = 1$ or 0 according to $i \in \alpha$ or $i \notin \alpha$. Obviously, we have,

$$\sum_{\alpha \in V(i)} a(\alpha, i) = \nu_i, \qquad \sum_{i \in V^{-1}(\alpha)} a(\alpha, i) = f(\alpha).$$

§1. **Asymptotics of the two sums. Definition of a_r.** Let us consider the following typical term of the first kind,

$$\bigotimes_{i,j} \varLambda^{a_{i,j}}(X_i), \quad 1 \leq i \leq k, \quad 1 \leq j \leq \mu.$$

Put $m_i = \sum_{1 \leq j \leq \mu} a_{i,j}$ and put $X^{m_i} = \bigotimes_{1 \leq j \leq \mu} \varLambda^{a_{i,j}}(X_i)$. Let Z_+ denote the set of all nonnegative integers. For each vector $(m_1, \cdots, m_k) \in Z_+^k$, there are then

$$\prod_{1 \leq i \leq k} \binom{\mu}{m_i}$$

such terms. Thus the dimension of the space of invariants of terms of the first type is

$$\sum_{m_1 + \cdots + m_k = r} \binom{\mu}{m_1} \cdots \binom{\mu}{m_k} \cdot [X_1^{m_1} \otimes \cdots \otimes X_k^{m_k}]$$

$$= \sum_{m_1 + \cdots + m_k = r} \frac{\mu^r}{m_1! \cdots m_k!} [\bigotimes_{\alpha \in S} \alpha^{m^F(\alpha)}] + O(\mu^{r-1}), \quad \text{where} \quad \mu \gg r.$$

Here the product $\otimes$ extends over all the vertices and for each vertex $\alpha \in S$, we have

$$m^F(\alpha) = m_{\alpha_1} + \cdots + m_{\alpha_{f(\alpha)}}, \qquad V^{-1}(\alpha) = \{X_{\alpha_1}, \cdots, X_{\alpha_{f(\alpha)}}\},$$

see [2] for more details. We recall from [2] that

$$[\otimes_{\alpha \in S} \alpha^{m^F(\alpha)}] = \prod_{\alpha \in S} [\alpha^{m^F(\alpha)}] = \prod_{\alpha \in S} c(m^F(\alpha)),$$

where

$$c(0) = 1, \ c(n) = 0 \text{ for } n \text{ odd, and } c(2t) = t^{-1} \cdot \binom{2t}{t-1} \text{ for } t > 0.$$

As a result, we have

$$\sum [\text{first kind term}] = a_r \mu^r + O(\mu^{r-1}), \qquad \mu \gg r,$$

where

$$a_r = \sum_{m_1 + \cdots + m_k = r} \frac{1}{m_1! \cdots m_k!} \prod_{\alpha \in S} c(m^F(\alpha)), \qquad m_j \geq 0.$$

For the second sum, the determination of its exact value is difficult, but a rough estimation of its value is easy. In fact, the second sum is $O(\mu^{r-1})$ for $\mu \gg r$. From this we obtain

$$\dim \Lambda^r(\mu F)^G = a_r \mu^r + O(\mu^{r-1}), \qquad \mu \gg r, \quad a_r \text{ as above.}$$

§2. Some combinatorial notation. For later purposes, we need some notation to deal with our combinatorics. A subpolyhedron $Q = X_{i_1} + \cdots + X_{i_\nu}$ of $P = X_1 + \cdots + X_k$ is called a *stable picture* if each vertex of Q belongs to an even number of faces of Q. A number of such examples can be found in [2]. The set of all stable pictures in P will be denoted by $\Pi = \Pi(P)$ and its cardinality will be denoted by $n(P) = |\Pi|$. For a stable picture Q of P, the number of faces in Q will be denoted be $\sharp(Q)$.

Example (see [2]). For the octahedron P, $n(P) = 16$, $\sharp(P) = 8$, $\sharp(\Pi) = \sharp(\Pi') = 4$, $\sharp(\Pi_i^{(1)}) = \sharp(\Pi_j^{(2)}) = 4$, $1 \leq i, j \leq 6$, and $\sharp(\phi) = 0$.

Clearly, ϕ is always a stable picture and $\sharp(\phi) = 0$. As a result $n(P) > 0$ holds for any polyhedron P.

§3. Entropy. The value of a_r is called the *total entropy*. It is the aim of the present work to estimate this quantity.

The vector $\vec{m} = (m_1, \cdots, m_k) \in Z_+^k$ is viewed as a Z_+-valued function defined on the set F of faces of P and is called a *distribution*. The *weight* $|\vec{m}|$ of $\vec{m}$ is defined to be the sum $m_1 + \cdots + m_k$. The individual term

$$\frac{1}{m_1! \cdots m_k!} \prod_{\alpha \in S} c(m^F(\alpha)),$$

where

$$m^F(\alpha) = m_{\alpha_1} + \cdots + m_{\alpha_{f(\alpha)}} \quad \text{and} \quad V^{-1}(\alpha) = \{X_{\alpha_1}, \cdots, X_{\alpha_{f(\alpha)}}\}$$

is called a *local entropy at the distribution* $\vec{m}$ and is denoted by $a(\vec{m})$.

A distribution $\vec{m}$ is called *admissible* if and only if $m^F(\alpha)$ is even for each $\alpha \in S$. The set of all admissible distributions will be denoted by A. Thus, $A \subset Z_+^k$. For a stable picture Q, a distribution $\vec{m} = (m_1, \cdots, m_k)$ is called Q-*admissible* if and only if, m_i is odd when $X_i \in Q$ and m_i is even when $X_i \notin Q$. For example, $\vec{m}$ is ϕ-admissible if and only if all the m_i's

are even.   The set of all Q-admissible distributions is denoted by $A(Q)$.   It is then easy to see that $A = \coprod_{Q \in \Pi} A(Q)$ (disjoint union with Q ranging over all the stable pictures including ϕ).   For a stable picture Q, we will define a sequence $\{a_r^Q\}$ by the formula

$$a_r^Q = \sum_{\vec{m} \in A(Q), |\vec{m}| = r} a(\vec{m}), \qquad \text{where } a(\vec{m}) \text{ is defined above.}$$

Evidently, we have

Theorem. $a_r = \sum_{Q \in \Pi} a_r^Q$.

If $\vec{m}$ is not admissible, $a(\vec{m}) = 0$ and we can ignore it.   Otherwise, $a(\vec{m})$ is positive and is a product involving factorials.   Thus, we can extend it (as a function of $\vec{m}$) to a real analytic function of $\vec{x}$ defined on $\mathbf{R}_+^k$ by using Γ-functions.   Let this extension be denoted by $f(\vec{x})$.   Namely, (1) $f(\vec{x})$ is real analytic on $\mathbf{R}_+^k$ and (2) $f(\vec{m}) = a(\vec{m})$ if $a(\vec{m}) \neq 0$.   To be precise, we have

$$f(\vec{x}) = \prod_{\alpha \in S} \frac{2\Gamma(x^F(\alpha))}{\Gamma\left(\dfrac{x^F(\alpha)}{2}\right)\Gamma\left(\dfrac{x^F(\alpha)}{2} + 2\right)} \prod_{1 \leq i \leq k} \frac{1}{\Gamma(x_i + 1)},$$

where

$$x^F(\alpha) = x_{\alpha_1} + \cdots + x_{\alpha_{f(\alpha)}} \quad \text{and} \quad V^{-1}(\alpha) = \{X_{\alpha_1}, \cdots, X_{\alpha_{f(\alpha)}}\}.$$

The following result is then clear.

Proposition. $a_r = \sum_{\vec{m} \in A, |\vec{m}| = r} f(\vec{m})$. $a_r^Q = \sum_{\vec{m} \in A(Q), |\vec{m}| = r} f(\vec{m})$, where Q is a stable picture.

It is not easy to compute the values of a_r for even $r \geq 0$.   To do so would require repeated computations of

$$\prod_\alpha c(m^F(\alpha)) \cdot \prod_i (m_i!)^{-1}$$

and then to sum up the results.   In special cases, values of a_r were obtained by computer calculation using Basic on a *PC*.   To obtain results for big r and various P, we would need a much bigger computer and a more sophisticated program language.

§ 4.   **Statement of the main result.**   We end the present note with a statement of our main result.

Main theorem.   *Let* $M = \sum_{1 \leq i \leq k} 2^{\nu_i}$ *and let* $\vec{\omega} = (\omega_1, \cdots, \omega_k)$ *with* $\omega_i = 2^{\nu_i} \cdot M^{-1}$, $1 \leq i \leq k$.   *Let* Q *be any stable picture.   Then* a_r^Q *is asymptotically equal to*

$$2^{-k} \cdot 4^m \cdot (2\pi)^{-m/2} \cdot \prod_{\alpha \in S} \omega^F(\alpha)^{-3/2} \cdot \frac{M^r}{r^{3m/2} \cdot r!}.$$

The details of the proof together with some corollaries will be given in the next installment, see [3].

References

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