50. The Differentiable Pinching Problem and the Diffeotopy Theorem

By Yoshihiko SUYAMA*)
Department of Mathematics, College of General Education, Tohoku University

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1. Introduction. Let (M, g) be a complete, simply connected riemannian manifold of dimension n. The purpose of this note is to announce the following theorem and to state the main idea of the proof. A detailed account will be published elsewhere [9].

Theorem 1. *Let (M, g) be a 0.681-pinched riemannian manifold. Then M is diffeomorphic to the standard sphere S^n.*

We say that (M, g) is a δ-pinched riemannian manifold, if the sectional curvature K satisfies $\delta \leqq K \leqq 1$.

A riemannian manifold with $\frac{1}{4} < K \leqq 1$ is homeomorphic to the standard sphere by the sphere theorem [1, 5]. The discovery of exotic spheres by Milnor gave rise to the question when the conclusion in the sphere theorem could be replaced by diffeomorphism. We call it the differentiable pinching problem. For the first time, Gromoll [2], Calabi and Shikata [7] gave some results on this problem. Later on, their results were improved as follows:

Theorem (Sugimoto-Shiohama [8]). *Let (M, g) be a 0.87-pinched riemannian manifold. Then M is diffeomorphic to the standard sphere.*

Theorem (Im Hof-Ruh [4]). *There exists a decreasing sequence δ_n with limit $\delta_n \to 0.68$ as n tends to infinity such that the following assertion holds:*

If (M, g) is a δ_n-pinched riemannian manifold of dimension n, and $\mu: G \times M \to M$ is an isometric action of the Lie group G on M, then

(1) *there exists a diffeomorphism $F: M \to S^n$,*

(2) *there exists a homomorphism $\psi: G \to O(n+1)$ such that*

(3) *$\psi(g) = F \circ \mu(g, \cdot) \circ F^{-1}$ for all $g \in G$.*

We are interested in a pinching number independent of dimension of M. In Im Hof-Ruh's result, if we take the number δ independent of dimension of M, then δ becomes considerably large, i.e., $\delta = 0.98$ for $n > 5$. Our pinching number 0.681 is almost same as the number limit $\delta_n = 0.68$ given by Im Hof-Ruh. But their numbers are determined by quite different equations from each other.

2. Ideas. Sugimoto-Shiohama's beginning idea was as follows: A complete, simply connected δ-pinched riemannian manifold M^n is diffeomorphic to the standard sphere S^n if a diffeomorphism $f: S^{n-1} \to S^{n-1}$,

*) Current address: Department of Applied Mathematics, Faculty of Science, Fukuoka University.

which is naturally defined for δ-pinched riemannian maifold M [see § 4], is diffeotopic to the identity map. We shall call this the diffeotopy idea. So, their important idea was to construct such a diffeotopy, and to find an explicit estimate for δ to guarantee such a diffeotopy. On the other hand, the main idea in a series of papers Ruh [7], Grove-Karcher-Ruh [3] and Im Hof-Ruh, was to lead from a connection with small curvature on the stabilized tangent bundle E of M to a flat connection on this bundle. This first connection with small curvature on the bundle was defined with relation to the pinching number δ. We shall call this the flat connection idea. Using the resulting flat connection, they defined a generalized Gauss map $G: M \to S^n$, which gave a diffeomorphism. So, the problems in this case were how to construct a flat connection from the connection with small curvature on E, and how to find an explicit estimate for δ in order that the Gauss map could be diffeomorphism.

It has been considered that two ideas of diffeotopy and flat connection are independent of each other [10, Geodesics (178, E)]. But we can show that the flat connection idea is based on the almost same consideration as the diffeotopy idea.

We use the diffeotopy idea in the proof of Theorem 1, that is, we find a sufficient condition that the diffeomorphism $f: S^{n-1} \to S^{n-1}$ is diffeotopic to the identity map. But our diffeotopy is constructed in a different way from Sugimoto-Shiohama's. Our idea is as follows. $f: S^{n-1} \to S^{n-1}$ is homothetically extended to a diffeomorphism $F: R^n - \{0\} \to R^n - \{0\}$. The restriction of the differential dF to S^{n-1} becomes a map of S^{n-1} into the space $M(n, R)$ of $n \times n$-matrices. We approximate $dF: S^{n-1} \to M(n, R)$ by a map $\alpha: S^{n-1} \to SO(n, R)$. For a differentiable map $\alpha: S^{n-1} \to SO(n, R)$, we denote by α_x the matrix corresponding to $x \in S^{n-1}$. And then, we construct the diffeotopy by joining α_x to a constant matrix by geodesic in $SO(n, R)$ for each $x \in S^{n-1}$.

3. Diffeotopy theorem. In this section, we state exactly our diffeotopy theorem.

Let S^{n-1} be the standard sphere with curvature 1. Let $f: S^{n-1} \to S^{n-1}$ be a diffeomorphism. We put $F(tx) = tf(x)$ for $t > 0$. We define a norm of differential of $\alpha: S^{n-1} \to SO(n, R)$ by

$$\|d\alpha\| = \max\{\|(d_X\alpha)U\| \mid X \in T_x(S^{n-1}) \text{ and } U \in R^n \text{ with } \|X\| = \|U\| = 1\},$$

where $\|X\|$ denotes the euclidian norm of X. Similarly, the norm of a map $A: S^{n-1} \to M(n, R)$ is defined by

$$\|A\| = \max\{\|A_x U\| \mid x \in S^{n-1} \text{ and } U \in R^n \text{ with } \|U\| = 1\}.$$

We say that f is diffeotopic to the identity map of S^{n-1}, if there exists a differentiable map $H: [0, 1] \times S^{n-1} \to S^{n-1}$ satisfying the following (1) and (2):

(1) $H(1, x) = f(x)$ and $H(0, x) = x$.

(2) The map $H_t = H(t, \cdot)$ is diffeomorphism of S^{n-1} for each t.

And we say that $\alpha: S^{n-1} \to SO(n, R)$ is an approximation of df on S^{n-1}, if there exist real numbers C and N and they satisfy the following (1), (2), (3) and (4):

(1) $N<1$. (2) $\alpha_x(x)=(d_xF)(x)$ for $x\in S^{n-1}$.

(3) $\|\alpha-dF\|\leqq C$. (4) $\|d\alpha\|\leqq N$.

Note that the inequality $C\leqq N$ always holds. For the approximation α of df, we define a positive function $P(t)$ for $t\in[0,\pi]$ by

$$P(t)^2=C_2^2\left(\frac{\sin(Nt/2)}{\sin(N\pi/2)}\right)^2+C_3^2\left(\frac{\sin(Nt)}{\sin(N\pi)}\right)^2+2C_2C_3\frac{\sin(Nt)}{\sin(N\pi)}\psi(t),$$

where $C_2=(N-C)/2$, $C_3=(N+C)/2$ and $\psi(t)$ is defined by

$$\psi(t)=\frac{\sin(Nt/2)}{\sin(N\pi/2)}\,(0\leqq t\leqq t_0),\qquad -\frac{\sin(Nt/2)}{\sin(N\pi/2)}\cos\left(\frac{3}{2}N(\pi-t)\right)\,(t_0\leqq t\leqq t_1),$$

$$-\frac{t}{\pi}\cos\left(\frac{3}{2}N(\pi-t)\right)\,(t_1\leqq t\leqq\pi).$$

Above t_0 and t_1 are given by $\cos(3N(\pi-t_0)/2)=-1$ and $\cos(3N(\pi-t_1)/2)=0$ respectively.

Theorem 2. *Let $f: S^{n-1}\to S^{n-1}$ be a diffeomorphism. Suppose there exist an approximation α of df such that $P(t)<1$ for $t\in[0,\pi]$. Then f is diffeotopic to the identity map.*

4. The estimates of $\|\alpha-dF\|$ and $\|d\alpha\|$. Let $\delta>1/4$. The manifold M is homeomorphic to the standard sphere by the sphere theorem. In particular, we use the following properties. Let q_0 and q_1 be a pair of points with maximal distance $d(q_0,q_1)$ on M, where d denotes the distance function induced by the riemannian metric. Set $X(p)=d(q_0,p)-d(q_1,p)$, and define $C=X^{-1}(0)$, $M_0=X^{-1}((-\infty,0])$ and $M_1=X^{-1}([0,\infty))$. The exponential maps $\exp_0$ and $\exp_1$ with centers at q_0 and q_1 respectively are bijective maps if restricted to a ball of radius π. C is diffeomorphic to S^{n-1}. Let S_0 and S_1 denote unit spheres in tangent spaces of points q_0 and q_1. Then the diffeomorphism $f: S_0\to S_1$ is defined by requiring $\exp_0(tx)$ and $\exp_1(tf(x))$ to coincide for some $t=t(x)$ satisfying $\pi/2\leqq t(x)\leqq\pi/(2\sqrt{\delta})$. Note that the point of intersection lies on C. We denote by $q(x)$ the point $\exp_0(t(x)x)=\exp_1(t(x)f(x))\in C$.

We fix suitable orthonormal bases of tangent spaces $T_0(M)$ of q_0 and $T_1(M)$ of q_1, respectively. Put $\tau^0(x,t)=\exp_0(tx)$ and $\tau^1(f(x),t)=\exp_1(tf(x))$ for $x\in S_0$. For a vector $X\in T_{q(x)}(C)$, we denote by X_0 (resp. X_1) the vector of $T_0(M)$ (resp. $T_1(M)$) obtained by parallel translation of $X-g(X,\dot\tau^0(x))\dot\tau^0(x)\in T_{q(x)}(C)$ (resp. $X-g(X,\dot\tau^1(f(x)))\dot\tau^1(f(x))\in T_{q(x)}(C)$) along a geodesic $\tau^0(x,t)$ (resp. $\tau^1(f(x),t)$). Then our approximation $\alpha_x\in SO(n,R)$ of df_x for $x\in S_0$ is defined as follows:

(1) $\alpha_x(X_0)=X_1$ for $X\in T_{q(x)}(C)$, (2) $\alpha_x(x)=f(x)$,

where X_i $(i=0,1)$ denotes the component vector with respect to the base. Then we have the estimate of $\|\alpha-dF\|$ by using Levi-Civita connection of M. An estimate of $\|dF-\alpha\|$ was also given by Sugimoto-Shiohama. But, we can give the estimate sharper than it. Furthermore, on the above construction of diffeotopy we use the estimate in a quite different way from that of Sugimoto-Shiohama.

On the other hand, with the estimate of $\|d\alpha\|$ we use the stabilized tangent bundle E and the connection ∇ with small curvature of E due to Ruh. Let P be the associated principal bundle of E with structure group $O(n+1, R)$. We take a horizontal lift $u^0(x, t)$ (resp. $u^1(f(x), t)$) in P of $\tau^0(x, t)$ (resp. $\tau^1(f(x), t)$) under a suitable initial condition $u^0(x, 0)$ (resp. $u^1(f(x), 0)$). Then there exists $b_x \in O(n+1, R)$ such that $u^0(x, t(x))b_x = u^1(f(x), t(x))$. The scale $\|db\|$ is determined by the norm of curvature of the connection ∇. Furthermore, by investigating the difference between α_x and b_x we have the estimate of $\|d\alpha\|$.

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