46. *Newforms of Half-integral Weight and the Twisting Operators*

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0. In the papers [4] and [5], we report some trace relations of the twisting operators on the space of cusp forms of half-integral weight $S(k+1/2, N, \chi)$ and on the Kohnen subspace $S(k+1/2, N, \chi)_K$. In this paper, we shall use these trace relations of the twisting operators in order to decompose the spaces $S(k+1/2, N, \chi)$ and $S(k+1/2, N, \chi)_K$ into nice subspaces, i.e., the space of "*newforms*" which correspond in one to one way to a system of eigen-values for Hecke operators. For simplicity of statements, we treat only the case of the Kohnen subspace of level $4p^m$, weight $k+1/2$ and a character χ, where p is an odd prime number, $2\le m \in \boldsymbol{Z}$, $2\le k\in \boldsymbol{Z}$, and χ is an even character modulo $4p^m$ such that $\chi^2=1$. More general results and details will appear in [6].

1. We keep to the notations and the assumptions in [4]. Let $\psi=\left(\frac{}{p}\right)$ be the quadratic residue symbol. Since the twisting operator R_ψ for ψ satisfies the identity $R_\psi^3=R_\psi$ as operators, R_ψ is a semi-simple operator and the eigen values of R_ψ are 1, 0, or -1. We denote the σ-eigen subspace of $\tilde{S}=\tilde{S}(p^m, \chi)=S(k+1/2, 4p^m, \chi)_K$, $\sigma=0$, 1, or -1, by: $\tilde{S}^0=\tilde{S}^0(p^m, \chi)$ if $\sigma=0$ and $\tilde{S}^\pm=\tilde{S}^\pm(p^m, \chi)$ if $\sigma=\pm 1$. Then we have $\tilde{S}=\tilde{S}^0\oplus\tilde{S}^+\oplus\tilde{S}^-$ and moreover

$$\tilde{S}^0=\mathrm{Ker}\,(R_\psi\,|\,\tilde{S})=\left[S\left(k+1/2,\,4p^{m-1},\,\chi\left(\frac{p}{}\right)\right)_K\right]^{(p)}.$$

Here, we put $[S(k+1/2, 4p^m, \chi)_K]^{(p)}=\{f(pz)\,|\,f\in S(k+1/2, 4p^m, \chi)_K\}$. This equality follows from the following lemma.

Lemma. *Let N be a positive integer divisible by 4, χ an even character modulo N, and l an odd prime divisor of N. If a function f on $\mathfrak{H}$ is satisfies the following two conditions*:

(i) $f(z)=f(z+1)$ *for all* $z\in\mathfrak{H}$, (ii) $f(lz)\in S(k+1/2, N, \chi)$,
then we have

$$f\in S\left(k+1/2,\,N/l,\,\chi\left(\frac{l}{}\right)\right).$$

In particular, if the conductor of $\chi\left(\frac{l}{}\right)$ does not divide N/l, then $f=0$.

Remark. This lemma is an analogy of the Theorem 4.6.4 of [1].

From $\tilde{T}(n^2)R_\psi=R_\psi\tilde{T}(n^2)$ ([5, Prop. (1.7)]), we have the following formulae:

$$(1)\quad \mathrm{tr}(R_\psi \tilde{T}(n^2)\,|\,S(k+1/2, 4p^m, \chi)_K) = \mathrm{tr}(\tilde{T}(n^2)\,|\,\tilde{S}^+(p^m,\chi)) - \mathrm{tr}(\tilde{T}(n^2)\,|\,\tilde{S}^-(p^m,\chi)),$$

$$(2)\quad \mathrm{tr}(\tilde{T}(n^2)\,|\,S(k+1/2, 4p^m, \chi)_K) = \mathrm{tr}(\tilde{T}(n^2)\,|\,\tilde{S}^0(p^m,\chi)) + \mathrm{tr}(\tilde{T}(n^2)\,|\,\tilde{S}^+(p^m,\chi)) + \mathrm{tr}(\tilde{T}(n^2)\,|\,\tilde{S}^-(p^m,\chi)),$$

and

$$(3)\quad \mathrm{tr}\left(\tilde{T}(n^2)\,\Big|\,S\left(k+1/2, 4p^{m-1}, \chi\left(\frac{p}{\ }\right)\right)_K\right) = \mathrm{tr}(\tilde{T}(n^2)\,|\,\tilde{S}^0(p^m,\chi)).$$

From [3, Theorem] and [4, Theorem], we can rewrite the left hand side of the formulae (1)–(3) as follows.

$$(4)\quad \mathrm{tr}(\tilde{T}(n^2)\,|\,S(k+1/2, 4p^m, \chi)_K) - \mathrm{tr}\left(\tilde{T}(n^2)\,\Big|\,S\left(k+1/2, 4p^{m-1}, \chi\left(\frac{p}{\ }\right)\right)_K\right)$$
$$= \mathrm{tr}(T(n)\,|\,S(2k, p^m)) - \mathrm{tr}(T(n)\,|\,S(2k, p^{m-1})) + \chi_p(-n)\,\mathrm{tr}([W(p^{\hat{m}})]_{2k} T(n)\,|\,S(2k, p^{\hat{m}})),$$

and

$$(5)\quad \mathrm{tr}(R_\psi \tilde{T}(n^2)\,|\,S(k+1/2, 4p^m, \chi)_K) = \left(\frac{-1}{p}\right)^k \chi_p(n)\,\mathrm{tr}([W(p^{\tilde{m}})]_{2k} T(n)\,|\,S(2k, p^{\tilde{m}})).$$

Here, $\hat{m}$ (resp. $\tilde{m}$) is the greatest even (resp. odd) integer x such that $x \leq m$.

Therefore, we have

$$(6)\quad 2\,\mathrm{tr}(\tilde{T}(n^2)\,|\,\tilde{S}^\pm(p^m,\chi)) = \mathrm{tr}(T(n)\,|\,S(2k, p^m)) - \mathrm{tr}(T(n)\,|\,S(2k, p^{m-1})) + \chi_p(-n)\,\mathrm{tr}([W(p^{\hat{m}})]_{2k} T(n)\,|\,S(2k, p^{\hat{m}})) \pm \left(\frac{-1}{p}\right)^k \chi_p(n)\,\mathrm{tr}([W(p^{\tilde{m}})]_{2k} T(n)\,|\,S(2k, p^{\tilde{m}})).$$

2. For $i, j \in Z$ $(2 \leq j < i)$, $\tilde{S}^\pm(p^i, \chi)$ contains $\tilde{S}^\pm(p^j, \chi)$. Then for $m \geq 3$, we can define the orthogonal complement $\tilde{S}^\pm(p^m, \chi)^0$ of $\tilde{S}^\pm(p^{m-1}, \chi)$ in $\tilde{S}^\pm(p^m, \chi)$.

Now, we know the following relation:

$$\mathrm{tr}([W(p^m)]_{2k} T(n)\,|\,S(2k, p^m)) = \sum_{a=0}^{[m/2]} \mathrm{tr}([W(p^{m-2a})]_{2k} T(n)\,|\,S^0(2k, p^{m-2a})).$$

Here, $S^0(2k, p^{m-2a})$ denotes the subspace of $S(2k, p^{m-2a})$ spanned by all newforms in $S(2k, p^{m-2a})$. From this relation, we have

Proposition. (7) *For any odd integer* $m \geq 3$,

$$\mathrm{tr}(\tilde{T}(n^2)\,|\,\tilde{S}^\pm(p^m,\chi)^0) = \frac{1}{2}\left\{\mathrm{tr}(T(n)\,|\,S^0(2k, p^m)) \pm \left(\frac{-1}{p}\right)^k \chi_p(n)\,\mathrm{tr}([W(p^m)]_{2k} T(n)\,|\,S^0(2k, p^m))\right\}.$$

(8) *For any even integer* $m \geq 4$,

$$\mathrm{tr}(\tilde{T}(n^2)\,|\,\tilde{S}^\pm(p^m,\chi)^0) = \frac{1}{2}\{\mathrm{tr}(T(n)\,|\,S^0(2k, p^m)) + \chi_p(-n)\,\mathrm{tr}([W(p^m)]_{2k} T(n)\,|\,S^0(2k, p^m))\}.$$

For $m \geq 3$, we define (cf. [2])

$$S_I = S_I(2k, p^m) := \{f \in S^0(2k, p^m)\,|\,f|W = f,\ f|RW = f|R\},$$
$$S_{II} = S_{II}(2k, p^m) := \{f \in S^0(2k, p^m)\,|\,f|W = f,\ f|RW = -f|R\},$$
$$S_{II_\psi} = S_{II_\psi}(2k, p^m) := \{f \in S^0(2k, p^m)\,|\,f|W = -f,\ f|RW = f|R\},$$

$$S_{III}=S_{III}(2k, p^m):=\{f\in S^0(2k, p^m)\,|\,f|W=-f,\ f|RW=-f|R\},$$
where $R=R_\psi$ and $W=W(p^m)$.

We denote by $S^{0,\pm}(2k, p^m)$ ± 1-eigen subspace of $S^0(2k, p^m)$ on the operator W. Furthermore we denote by $H(p^m)$ the restricted Hecke algebra which is defined in [3, p. 543]. Then we have the following.

Theorem. *For $m\geq 3$, we have the following isomorphisms as $H(p^m)$-modules.*

$$(9)\qquad \tilde{S}^{\pm}(p^m, \chi)^0\cong S^{0,\pm\left(\frac{-1}{p}\right)^k}(2k, p^m)\qquad \text{if } \chi=\left(\frac{1}{\ }\right) \text{ and } m \text{ is odd.}$$

$$(10)\qquad \tilde{S}^{\pm}(p^m, \chi)^0\cong S^{0,+}(2k, p^m)\qquad \text{if } \chi=\left(\frac{1}{\ }\right) \text{ and } m \text{ is even.}$$

$$(11)\qquad \tilde{S}^{\pm}(p^m, \chi)^0\cong \frac{1}{2}\left(1\pm\left(\frac{-1}{p}\right)^k\right)\{S_I\oplus S_{II_\psi}\}\oplus\frac{1}{2}\left(1\mp\left(\frac{-1}{p}\right)^k\right)\{S_{II}\oplus S_{III}\},$$
$$\text{if } \chi=\left(\frac{p}{\ }\right) \text{ and } m \text{ is odd.}$$

$$(12)\qquad \tilde{S}^{\pm}(p^m, \chi)^0\cong \frac{1}{2}\left(1+\left(\frac{-1}{p}\right)\right)\{S_I\oplus S_{II_\psi}\}\oplus\frac{1}{2}\left(1-\left(\frac{-1}{p}\right)\right)\{S_{II}\oplus S_{III}\},$$
$$\text{if } \chi=\left(\frac{p}{\ }\right) \text{ and } m \text{ is even.}$$

From these isomorphisms, we have a strong "multiplicity 1 *theorem" for the space* $\tilde{S}^{\pm}(p^m, \chi)^0$.

We shall call the space $\tilde{S}^{\pm}(p^m, \chi)^0$ the space of *newforms* in $S(k+1/2, 4p^m, \chi)_K$. This naming is justified by the above theorem. We can define the space of "*newforms*" for more general case. See [6] for details.

References

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