44. On 3-Connected 10-Dimensional Manifolds*)

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1. Introduction. In [4], the author gave a complete homotopy classification of 2-connected smooth 8-manifolds with vanishing 4th homology groups as an application of the homotopy classification theory of primary manifolds. In this paper, as another application of it, we classify 3-connected smooth 10-manifolds M satisfying the following hypotheses:

(H1) $H_4(M)$ is torsion free.

(H2) The tangent bundle of M is trivial on its 4-skeleton.

It is easily seen that (H2) is equivalent to

(H2′) The tangent bundle of M is stably trivial.

Thus, our classification is that of 3-connected 10-dimensional π-manifolds with torsion free homology groups. Henthforth, manifolds are smooth, oriented, connected, and closed unless mentioned explicitly. Homotopy equivalences and diffeomorphisms are orientation preserving. The proofs of the theorems are given briefly.

Theorem 1. *Let M be a 3-connected 10-manifold satisfying* (H1), (H2). *Then, there exists a connected sum decomposition*

$$M=M_1\#(S^5\times S^5)\#\cdots\#(S^5\times S^5),$$

where M_1 is a 3-connected 10-manifold satisfying (H1), (H2) *and* $H_5(M_1)=0$. *The decomposition is unique up to diffeomorphism, that is, if there exists another decomposition as above by M_1' and $S^5\times S^5$'s, then M_1, M_1' must be diffeomorphic and the numbers of $S^5\times S^5$ are equal.*

Let $\mathcal{H}(p+q+1, r, q)$ be the set of the handlebodies obtained by gluing q-handles, r in number, to a $(p+q+1)$-disk. In the following theorem, the symmetric bilinear form $\psi: H^4(M)\times H^4(M)\to Z_2$ is defined by $\psi(x, y)=\langle S_q^2x_2\cup y_2, [M]_2\rangle$, where x_2, y_2, $[M_2]$ denote x, y, $[M]$ in Z_2 coefficient respectively. The type of M is defined by using ψ (cf. [2], [3]). So, it is a homotopy invariant of M and coincides with the type of the handlebody W of $\mathcal{H}(11, r, 6)$, $r=\operatorname{rank} H_4(M)$, bounded by M up to homotopy equivalence (cf. Theorem 8.3 of [2]). The following theorem completes our classification up to diffeomorphism $\operatorname{mod}\Theta_{10}$.

Theorem 2. *Let M, M' be 3-connected 10-manifolds satisfying the hypothesis* (H2) *and* $H_5(M)=H_5(M')=0$ *(so,* (H1) *is satisfied). Then, M, M' are diffeomorpic* $\operatorname{mod}\Theta_{10}$ *if and only if M, M' are homotopy equivalent. Such manifolds M with fixed* $\operatorname{rank} H_4(M)=r$ *can be completely classified up to homotopy equivalence, and hence up to diffeomorphism* $\operatorname{mod}\Theta_{10}$, *and the*

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independent representatives are given as follows, where s( ) denotes the connected sum of s copies of ( ):

(i) *Those of type* 0 *are given by*

$$r(S^6\times S^4),$$
$$A_1\#(r-1)(S^6\times S^4),$$

where A_1 is the S^4-bundle over S^6 with the characteristic element $1\in Z_2=\pi_5(SO_5)$ and admits a cross-section.

(ii) *There is no manifold of type* I *or of type* (0+I).

(iii) *Those of type* II *are given by*

$$cV_0,$$
$$V_1\#(c-1)V_0,$$

where $r=2c$, $V_i=\partial W(i)$, $i=0$, 1, and $W(i)\in\mathcal{H}(11,2,6)$ is determined by the invariant system $(H;\phi,\alpha)$ such that $\phi(e_1,e_2)=1\in Z_2=\pi_6(S^5)$, $\alpha(e_1)=\alpha(e_2)=i\in Z_2=\pi_5(SO_5)$ for the basis $\{e_1,e_2\}$ of H corresponding to the handles.

(iv) *Those of type* (0+II) *are given by*

$$a(S^6\times S^4)\#cV_0,$$
$$a(S^6\times S^4)\#V_1\#(c-1)V_0,$$
$$A_1\#(a-1)(S^6\times S^4)\#cV_0,$$

where $a+2c=r$ and $2c=\mathrm{rank}\,\psi$.

(For the definition and the properties of invariant systems of handlebodies, see [8]).

Remark. The manifolds V_0, V_1 of type II are not homotopy equivalent unlike 8-dimensional case. But, we can show that the suspensions of those are homotopy equivalent. This fact gives a counter example to Conjecture 4.14 of [9].

The following theorem means that for our manifolds, the homotopy classification is equivalent to the classification up to diffeomorphism mod Θ_{10}.

Theorem 3. *Let M, M' be 3-connected* 10*-manifolds satisfying* (H1), (H2). *If M, M' are homotopy equivalent, then in their decompositions by* Theorem 1 *with M_1, M_1' and $S^5\times S^5$'s respectively, M_1 is diffeomorphic to M_1' mod Θ_{10}, and hence M is diffeomorphic to M' mod Θ_{10}.*

2. Proofs of the theorems. Since the Kervaire invariant of a 4-connected 10-manifold vanishes, Theorem 1 is obtained by Theorem 3 of [2] and the Uniqueness Theorem of [5]. To prove Theorem 2, we need the following lemmas.

Lemma 4. *For $\pi_9(S^4)=\{E\nu'\circ\eta_7^2\}+\{\nu_4\circ\eta_7^2\}\cong Z_2+Z_2$, the homomorphism induced from the inclusion $i_*: \pi_9(S^4)\to\pi_9(S^4\cup_{\eta_4}D^6)$ satisfies $i_*(E\nu'\circ\eta_7^2)=0$, $i_*(\nu_4\circ\eta_7^2)\neq 0$.*

Proof. We study $\partial_*:\pi_{10}(S^4\cup_{\eta_4}D^6,S^4)\to\pi_9(S^4)$ of the exact sequence. By Theorem 2.1 of [6], the homomorphism $Q:\pi_5(S^4)\to\pi_{10}(S^4\cup_{\eta_4}D^6,S^4)$ which takes the relative Whitehead product with the orientation generator σ_6 of $\pi_6(S^4\cup_{\eta_4}D^6,S^4)\cong Z$ is surjective since $\pi_{10}(S^6)=0$. So, $\pi_{10}(S^4\cup_{\eta_4}D^6,S^4)$ is generated by $[\sigma_6,\eta_4]$. We have $\partial_*[\sigma_6,\eta_4]=[\partial_*\sigma_6,\eta_4]=[\eta_4,\eta_4]=\eta_4\circ[\iota_5,\iota_5]$. Since

$[\iota_5, \iota_5] \neq 0$ and $\pi_9(S^5) = \{\nu_5 \circ \eta_8\} \cong Z_2$, $[\iota_5, \iota_5]$ must be $\nu_5 \circ \eta_8$. Hence, $\partial_*[\sigma_6, \eta_4] = \eta_4 \circ \nu_5 \circ \eta_8 = E\nu' \circ \eta_7 \circ \eta_8 = E\nu' \circ \eta_7^2$ (cf. (5.9) of [7, p. 44]). Thus, $\pi_{10}(S^4 \cup_{\eta_4} D^6, S^4) = Z_2[\sigma_6, \eta_4]$ and we have the lemma.

Lemma 5. *Let* $\bar{i}_*: \pi_9(S^4)/\mathrm{Im}\, P \to \pi_9(S^4 \cup_{\eta_4} D^6)/i_*(\mathrm{Im}\, P)$ *be the homomorphism induced from* $i_*: \pi_9(S^4) \to \pi_9(S^4 \cup_{\eta_4} D^6)$, *where* $P: \pi_6(S^4) \to \pi_9(S^4)$ *is defined by* $P(x) = [x, \iota_4]$. *Then,* $\mathrm{Im}\, P = \mathrm{Ker}\, i_* = \{E\nu' \circ \eta_7^2\}$, *so* $i_*(\mathrm{Im}\, P) = 0$, *and* $\bar{i}_*$ *is injective.*

Proof. Since $\eta_3 \circ \nu_4 = \nu' \circ \eta_6$ and $\eta_5 \circ \nu_6 = 0$ by (5.9) of [7, p. 44], we have $E(E\nu' \circ \eta_7^2) = E(E\nu' \circ \eta_7 \circ \eta_8) = E(\eta_4 \circ \nu_5 \circ \eta_8) = \eta_5 \circ \nu_6 \circ \eta_9 = 0$. So, $P\eta_4^2 = E\nu' \circ \eta_7^2$ since $\{\eta_4^2\} = \pi_6(S^4) \xrightarrow{P} \pi_9(S^4) \xrightarrow{E} \pi_{10}(S^5)$ is exact. Hence the assertion is known from Lemma 4.

Lemma 6. *Let* $\lambda: \pi_5(SO_5) = S\pi_5(SO_4) \to \pi_9(S^4)/\mathrm{Im}\, P$ *be the homomorphism defined by* $\lambda(S\xi) = (J\xi)$ *which is independent of the choice of* ξ, *where* $S: \pi_5(SO_4) \to \pi_5(SO_5)$ *is induced from the inclusion. Let* $\bar{\lambda}: \pi_5(SO_5) \to \pi_9(S^4 \cup_{\eta_4} D^6)$ *be the homomorphism defined by* $\bar{\lambda} = \bar{i}_* \circ \lambda$. *Then,* λ, $\bar{\lambda}$ *are injective.*

Proof. Since S is surjective and the J-homomorphisms $J^{(2)}: \pi_5(SO_5) = Z_2 \to \pi_{10}(S^5) = Z_2$, $J^{(3)}: \pi_5(SO_4) = Z_2 + Z_2 \to \pi_9(S^4) = Z_2 + Z_2$ are isomorphic by Proposition 2.1 of [10], λ can be replaced by $J^{(2)}$ under the isomorphism $\bar{E}: \pi_9(S^4)/\mathrm{Im}\, P \cong \pi_{10}(S^5)$. So, λ is injective and hence $\bar{\lambda}$ by Lemma 5.

Now, Theorem 2 is obtained by Theorem 1′ of [2] and by the following.

Proposition 7. *Let* M *be a* 3-*connected* 10-*manifold with* $\mathrm{rank}\, H_4(M) = r$ *satisfying* (H2) *and* $H_5(M) = 0$. *Then, there exists a handlebody* $W \in \mathcal{H}(11, r, 6)$ *such that* $M = \partial W \bmod \Theta_{10}$. *Let* M', W' *be as above. Then, the following three are equivalent*:

(i) *M, M' are homotopy equivalent.*

(ii) *M, M' are diffeomorphic* $\bmod \Theta_{10}$.

(iii) *W, W' are diffeomorphic.*

Proof. Since M can be modified to a homotopy sphere by surgeries, by constructing conversely, we have the above handlebody W. We may show only that (i) induces (iii). Assume that M, M' are homotopy equivalent and let $(H; \phi, \alpha)$, $(H'; \phi', \alpha')$ be the invariant systems of W, W' respectively. Since ∂W, $\partial W'$ are homotopy equivalent, W, W' belong to the same type. Then, since λ, $\bar{\lambda}$ are injective by Lemma 6, we can easily see that $(H; \phi, \alpha)$, $(H'; \phi', \alpha')$ are isomorphic in each type by Theorems 1, 2, and 3 of [3]. Hence, by Theorem 2 of [8], W, W' are diffeomorphic. We note that there exists no handlebody of type I or type (0+I). Because, $\phi(x, x) = E\pi_*\alpha(x) = 0$ for any $x \in H$ since $\pi_*: \pi_5(SO_5) \to \pi_5(S^4)$ is trivial.

To prove Theorem 3, we use the splitting theorem of [1]. Let $M_2 = M_2' = (S^5 \times S^5) \# \cdots \# (S^5 \times S^5)$ and $f: M = M_1 \# M_2 \to M' = M_1' \# M_2'$ be a homotopy equivalence. Let $M' = N_1' \cup N_2'$, $N_1' \cap N_2' = S^9$, and $N_i' = M_i' - \mathrm{Int}\, D_i'^{10}$, $i = 1, 2$. Then, by Theorem 1.1 of [1], there exist the submanifolds N_1, N_2 of M and a map $h: (M, N_1, N_2, S) \to (M', N_1', N_2', S^9)$ which restrictions to respective manifolds are homotopy equivalences, such that $M = N_1 \cup N_2$, $S = N_1 \cap N_2 = \partial N_1 =$

∂N_2 and f is homotopic to h. Here, S is a homotopy 9-sphere but we can show that $S=S^9$. In fact, since N_1 is homotopy equivalent to N_1' and $M(\supset N_1)$ satisfies (H2), We can kill $H_4(N_1)$ by surgeries on N_1 so that $S=\partial N_1$ is the boundary of a contractible manifold. Thus, $M=\tilde{M}_1 \# \tilde{M}_2$ for $\tilde{M}_i=N_i \cup D_i^{10}$, $i=1, 2$. Then, by Theorem 1, $\tilde{M}_2=(S^5\times S^5)\#\cdots\#(S^5\times S^5) \bmod \Theta_{10}$ since $\tilde{M}_2$ is 4-connected and so $\tilde{M}_1=M_1 \bmod \Theta_{10}$. On the other hand, since $h|N_1$ can be extended to a homotopy equivalence of $\tilde{M}_1$ to M_1', $\tilde{M}_1$ has the homotopy type of M_1', and hence, $\tilde{M}_1=M_1' \bmod \Theta_{10}$ by Theorem 2. Thus, $M_1=M_1' \bmod \Theta_{10}$.

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