34. Construction of Certain Maximal p-ramified Extensions over Cyclotomic Fields

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(Communicated by Shokichi IYANAGA, M. J. A., June 12, 1990)

§1. **Introduction.** Let p and m be, respectively, a fixed odd prime number and a fixed integer with $(p, m)=1$ and let $k=\boldsymbol{Q}(\cos(2\pi/m))$ and $K_\infty=k(\mu_{p^\infty})$. Denote by Ω_p the maximal pro-p abelian extension over K_∞ unramified outside p. Its odd part Ω_p^- contains the field

$$C=K_\infty(\varepsilon^{1/p^\infty}\,|\,\text{all circular units } \varepsilon \text{ of } K_\infty).$$

The extension Ω_p^-/C is of very delicate nature, and for example, when $k=\boldsymbol{Q}$, it is closely related to the Vandiver conjecture at p. We shall give a system of generators for the extension Ω_p^-/C (except for its "ω_p-component") by using the theory of special units of F. Thaine [3].

§2. **Statement of the results.** Fix an *even* $\boldsymbol{Z}_p$-valued character χ of $\Delta_p=\mathrm{Gal}(k(\mu_p)/k)$, and let χ' be the *odd* character associated to χ, i.e., $\chi'=\omega_p\cdot\chi^{-1}$. Here, ω_p is the Teichmüller character of Δ_p. Since the Galois group Δ_p acts on the pro-p abelian groups $\mathrm{Gal}(\Omega_p^-/K_\infty)$ and $\mathrm{Gal}(C/K_\infty)$ in the usual manner, we can decompose them by the Δ_p-action. Let $\Omega_p(\chi')$ be the maximal intermediate field of Ω_p^-/K_∞ fixed by the ψ-components $\mathrm{Gal}(\Omega_p^-/K_\infty)(\psi)$ for all odd $\boldsymbol{Z}_p$-valued characters ψ of Δ_p except χ'. Define the intermediate field $C(\chi')$ of C/K_∞ similarly.

To give a system of generators of the extension $\Omega_p(\chi')/C(\chi')$, we have to recall from [2] and introduce some notations. For a while, we fix a natural number n and let $K_n=k(\mu_{p^{n+1}})$. For an abelian group A and an integer N, we abbreviate the quotient A/NA as A/N. Let M be any power of p. Regarding $(\boldsymbol{Z}/M)[\Delta_p]$ as a subring of $(\boldsymbol{Z}/M)[\mathrm{Gal}(K_n/\boldsymbol{Q})]$, we decompose $(\boldsymbol{Z}/M)[\mathrm{Gal}(K_n/\boldsymbol{Q})]$ by the Δ_p-action. Denote its χ-component by $\Lambda_{n,\chi,M}$. Let E_n and C_n be, respectively, the group of units and that of circular units of K_n. By a theorem on units in a Galois extension and that $[E_n : C_n]<\infty$, we see that there exists a Galois stable submodule C'_n of C_n such that C'_n is cyclic over the group ring $\boldsymbol{Z}[\mathrm{Gal}(K_n/\boldsymbol{Q})]$ and $[E_n : C'_n]<\infty$. In the following, assume that $\chi\neq$trivial ($\chi'\neq\omega_p$). Since $\chi\neq$trivial, the χ-component $(C'_n/M)(\chi)$ is free and cyclic over $\Lambda_{n,\chi,M}$ for any M. Let $p^{\delta(n,\chi)}$ be the exponent of $(E_n/C'_n)(p)(\chi)$, and we abbreviate $\Lambda_{n,\chi,p^{2\delta(n,\chi)}}$ as $\Lambda_{n,\chi}$. For an integer i, we denote by ζ_i a fixed primitive i-th root of unity. Let

$$\xi_n(1)=\prod_{i|mp^{n+1}}((1-\zeta_i)(1-\zeta_i^{-1}))^{a_i}$$

be a fixed generator of $(C'_n/p^{2\delta(n,\chi)})(\chi)$ over the group ring $\Lambda_{n,\chi}$, here a_i is an element of $\Lambda_{n,\chi}$. For a prime number l with $l\equiv1 \pmod{mp^{n+1}}$, define an element $\xi_n(l)$ of $K_n(\mu_l)$ by

$$\xi_n(l)=\prod_{i\mid mp^{n+1}}((1-\zeta_l\cdot\zeta_i)(1-\zeta_l\cdot\zeta_i^{-1}))^{a_i}.$$

Let N_l be the norm map from $K_n(\mu_l)$ to K_n. Since $l\equiv 1 \pmod{mp^{n+1}}$, we see that $N_l\xi(l)=1$. Hence, there exists an element $a_n(l)$ of $K_n(\mu_l)$ such that $\xi_n(l)=a_n(l)^{\sigma_l-1}$, σ_l being a fixed generator of $\mathrm{Gal}(K_n(\mu_l)/K_n)$. Put $\kappa_n(l)=N_l a_n(l)$, which is defined modulo $(K_n^\times)^{l-1}$. As in [3], [1] and [2], the elements $\kappa_n(l)$ play an important role. Consider prime numbers l such that $l\equiv 1$ $(\mathrm{mod}\ mp^{n+1})$ and $l\equiv 1 \pmod{p^{2\delta(n,\chi)}}$. Then, regarding $\kappa_n(l)$ as an element of $K_n^\times/p^{2\delta(n,\chi)}$, we denote by $\kappa_n^\chi(l)$ its χ-component. Also, call $\kappa_n^\chi(1)$ the χ-component of $\xi_n(1)$ $(\in K_n^\times/p^{2\delta(n,\chi)})$. Although we want to construct p-ramified extensions over $C(\chi')$ by using elements of the form $\kappa_n^\chi(l)^{1/p^{\delta(n,\chi)}}$, we have to impose some conditions on l to control ramifications. So, let $L_{n,\chi}$ be the set of all prime numbers l with $l\equiv 1 \pmod{mp^{n+1}}$, $l\equiv 1 \pmod{p^{2\delta(n,\chi)}}$ and such that l splits completely in $K_n(\kappa_n^\chi(1)^{1/p^{\delta(n,\chi)}})$. Then, our result is

Theorem. *If $\chi'\neq\omega_p$, then*

$$\Omega_p(\chi')=C(\chi')((\sigma\cdot\kappa_n^\chi(l))^{1/p^{\delta(n,\chi)}}\mid \forall n\geq 1,\ \forall l\in L_{n,\chi},\ \forall\sigma\in\mathrm{Gal}(K_n/\boldsymbol{Q})).$$

Remark. When $\chi'=\omega_p$, it is known that $\Omega_p(\omega_p)=C(\omega_p)$ if and only if the Iwasawa λ invariant of the cyclotomic $\boldsymbol{Z}_p$-extension over $k=\boldsymbol{Q}(\cos(2\pi/m))$ is zero. In particular, when $k=\boldsymbol{Q}$, $\Omega_p(\omega_p)=C(\omega_p)$.

§3. **Proof of Theorem.** The following lemma gives a prime ideal decomposition of the principal ideal $(\kappa_n^\chi(l))$.

Lemma 1 ([2, Lemma 3]). *Let l be a prime number with $l\equiv 1$ (mod mp^{n+1}) and λ be a prime ideal of K_n over l. Then, there exists a $\mathrm{Gal}(K_n/\boldsymbol{Q})$-equivariant isomorphism φ_λ of the multiplicative group $(O_{K_n}/l)^\times$ onto the abelian group $(\boldsymbol{Z}/(l-1))[\mathrm{Gal}(K_n/\boldsymbol{Q})]$ such that*

$$(\kappa_n(l))\equiv\varphi_\lambda(\kappa_n(1))\cdot\lambda \bmod (l-1)I$$

here I is the free abelian group of all ideals of K_n, and elements of the group ring act on I multiplicatively.

Proof of the inclusion $\supset$: For a prime number l in $L_{n,\chi}$, $\kappa_n^\chi(1)$ is a $p^{\delta(n,\chi)}$-th power in $(O_{K_n}/l)^\times$. Therefore, by Lemma 1, there exists an ideal $\mathfrak{a}$ of K_n such that $(\kappa_n^\chi(l))=\mathfrak{a}^{p^{\delta(n,\chi)}}$. So, we obtain the inclusion $\supset$.

To prove the reverse inclusion, it suffices to show that the system $\{\kappa_n^\chi(l)\mid n\geq 1,\ l\in L_{n,\chi}\}$ is "ample" in the following sense. Let V be the submodule of $K_\infty^\times\otimes(\boldsymbol{Q}_p/\boldsymbol{Z}_p)$ such that

$$\Omega_p^-=K_\infty(a^{1/p^n}\mid \text{all } a\otimes p^{-n}\in V).$$

The following exact sequence is well known (see e.g. [4]):

$$1\longrightarrow(\cup E_n)\otimes(\boldsymbol{Q}_p/\boldsymbol{Z}_p)\longrightarrow V\xrightarrow{\ f\ }\varinjlim A_n^+\longrightarrow 1\ (*),$$

here A_n is the p-part of the ideal class group of K_n and A_n^+ is its even part. Recall that the homomorphism f is defined as follows: For $a\otimes p^{-n}\in V$, the extension $K_\infty(a^{1/p^n})/K_\infty$ is unramified outside p. Then, since all primes of k above p are infinitely ramified in K_∞, there exists an ideal $\mathfrak{a}$ of K_s such that $(a)_{K_s}=\mathfrak{a}^{p^n}$ for some ideal $\mathfrak{a}$ of K_s for sufficiently large s. We define $f(a\otimes p^{-n})$ to be the class of $\mathfrak{a}$.

Since the group C_n of circular units of K_n is of finite index in E_n, we obtain the following exact sequence by decomposing $(*)$ by the Δ_p-action,

$$1 \longrightarrow ((\cup C_n) \otimes (\boldsymbol{Q}_p/\boldsymbol{Z}_p))(\chi) \longrightarrow V(\chi) \longrightarrow \varinjlim A_n(\chi) \longrightarrow 1.$$

Since $\Omega_p(\chi')$ is generated over K_∞ by $V(\chi)$, and so is $C(\chi')$ by $((\cup C_n) \otimes (\boldsymbol{Q}_p/\boldsymbol{Z}_p))(\chi)$, it suffices to prove that for each $n \geq 1$ and for each $\mathfrak{c} \in A_n(\chi)$, there exist $l \in L_{n,\chi}$ and $\alpha \in \boldsymbol{Z}[\mathrm{Gal}(K_n/\boldsymbol{Q})]$ such that

$$f((\alpha \cdot \kappa_n^\chi(l)) \otimes p^{-\delta(n,\chi)}) = \mathfrak{c}.$$

The proof of the inclusion $\subset$: Fix a natural number n. Since $(C'_n/p^{2\delta(n,\chi)})(\chi)$ is free over the group ring $\Lambda_{n,\chi}$ with a generator $\kappa_n^\chi(1)$, we identify $(C'_n/p^{2\delta(n,\chi)})(\chi)$ with $\Lambda_{n,\chi}$ by $\kappa_n^\chi(1) \leftrightarrow 1$. Define a Galois equivariant homomorphism ψ by

$$\psi : (E_n/p^{2\delta(n,\chi)})(\chi) \longrightarrow (C'_n/p^{2\delta(n,\chi)})(\chi) = \Lambda_{n,\chi}$$
$$\varepsilon \longrightarrow \varepsilon^{p^{\delta(n,\chi)}}.$$

Lemma 2 ([2, Theorem 5]). *For each ideal class* $\mathfrak{c} \in A_n(\chi)$, *there exist infinitely many prime ideals* $\lambda \in \mathfrak{c}$ *of degree one satisfying*

(1) $l = \lambda \cap \boldsymbol{Q} \equiv 1 \pmod{mp^{n+1}}, \bmod p^{2\delta(n,\chi)})$,

(2) $\alpha \cdot \varphi_\lambda|_{E_n} \equiv \psi \pmod{p^{2\delta(n,\chi)}}$ *for some* $\alpha \in \Lambda_{n,\chi}$,

(3) $\varphi_\lambda|_{E_n} \equiv \beta \cdot \psi \pmod{p^{2\delta(n,\chi)}}$ *for some* $\beta \in \Lambda_{n,\chi}$,

here, φ_λ *is the isomorphism in Lemma* 1.

Now, fix an ideal class $\mathfrak{c}$ in $A_n(\chi)$, and take a prime ideal λ in $\mathfrak{c}$ satisfying the conditions (1), (2) and (3) in Lemma 2. Then, by (3) and the definition of ψ, we see that

$$\varphi_\lambda(\kappa_n^\chi(1)) \equiv \beta \cdot \psi(\kappa_n^\chi(1)) \bmod p^{2\delta(n,\chi)} \equiv 0 \bmod p^{\delta(n,\chi)}.$$

This and (1) imply that $l = \lambda \cap \boldsymbol{Q} \in L_{n,\chi}$. By (2) and Lemma 1, we see that

$$(\alpha \cdot \kappa_n^\chi(l)) \equiv \alpha \cdot \varphi_\lambda(\kappa_n^\chi(1)) \cdot \lambda \bmod p^{2\delta(n,\chi)} I \equiv \psi(\kappa_n^\chi(1)) \cdot \lambda \bmod p^{2\delta(n,\chi)} I.$$

Hence, by the definition of ψ and the fact that the exponent of $A_n(\chi)$ is smaller than or equal to $p^{\delta(n,\chi)}$ ([1]), we obtain $f((\alpha \cdot \kappa_n^\chi(l)) \otimes p^{-\delta(n,\chi)}) = \mathfrak{c}$. This completes the proof of Theorem.

References

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