21. *Asymptotic Behavior of the Solution for an Elliptic Boundary Value Problem with Exponential Nonlinearity*

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§ 1. **Introduction and results.** We consider the elliptic eigenvalue problem

(1.1) $$-\Delta u=\lambda e^u \text{ (in } \Omega), \qquad u=0 \text{ (on } \partial\Omega)$$

for $u \in C^2(\Omega)\cap C^0(\bar{\Omega})$ and $\lambda \in \boldsymbol{R}_+ \equiv (0, +\infty)$, where $\Omega \subset \boldsymbol{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. We say that $\kappa \in \Omega$ is a core of Ω if it is a critical point of $k(x)=K(x, x)$, where $K(x, y)=G(x, y)+(1/2\pi)\log|x-y|$, $G=G(x, y)$ being the Green function: $-\Delta G=\delta(x-y)$, $G|_{x\in\partial\Omega}=0$. When Ω is simply-connected, cores are finite. Furthermore, a core is unique if Ω is convex. For these facts, see Friedman [3] for example. On the other hand, for each core $\kappa \in \Omega$ satisfying a generic constrain, a branch S^* of the solutions $\{(u, \lambda)\}$ for (1.1) is constructed by the method of singular perturbation such a way that u makes one-point blow-up at κ as $\lambda \downarrow 0$. This fact has been established by Weston [9], Mosley [6] and Wente [8].

In the present note we show that conversely each family of solutions makes finite-point blow-up for star-shaped Ω as $\lambda \downarrow 0$, unless it approaches to the trivial solution $u=0$ of (1.1) for $\lambda=0$. More precisely,

Theorem. If Ω is simply connected and the family of solutions $\{u\}$ of (1.1) accumulates as $\lambda \downarrow 0$ to $v=8\pi E_\kappa(x)$ in $W^{1,p}(\Omega)$ $(1<p<2)$ and in $C(\bar{\Omega}\backslash\{\kappa\})$, then $\kappa \in \Omega$ is a core and the function $E_\kappa=E_\kappa(x)$ solves $-\Delta E_\kappa=\delta(\kappa)$ and $E_\kappa|_{\partial\Omega}=0$.

Spruck [7] has studied a similar property for Sinh-Gordon equation in the rectangular domain $R\subset\boldsymbol{R}^2$. We are much inspired by his work, but the finiteness of a blow-up point does not follow from his argument for general domains. Our result extends to other semilinear eigenvalue problem in two-dimensional domains with exponentially-dominated nonlinearities, and details will be published elsewhere.

§ 2. **Outline of Proof.** The proof is divided into three parts:

Claim 1. When Ω is star-shaped, then $\Sigma \equiv \lambda \int_\Omega e^u dx$ is bounded as $\lambda \downarrow 0$.

Claim 2. If $\{\Sigma\}$ is bounded, then $\{u\}$ accumulates to a $v \in W^{1,p}(\Omega)\cap C^\infty(\bar{\Omega}\backslash\{\kappa_1, \cdots, \kappa_l\})$ for some finite points $\kappa_1, \cdots, \kappa_l \in \Omega$.

Claim 3. If $\{\kappa_1, \cdots, \kappa_l\}=\{\kappa\}$, we have $\Sigma\to 8\pi$ and $v=8\pi E_\kappa$ with some

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core κ, whenever Ω is simply connected.

Proof of Claim 1. This follows immediately from Rellich-Pohozaev's identity.

Proof of Claim 2. By means of Kaplan's argument, we can show that $\|u\|_{L^\infty_{\mathrm{loc}}(\Omega)} \in O(1)$ by $\Sigma \in O(1)$. This fact, together with the GNN property utilizing the Kelvin transformation (Gidas-Ni-Nirenberg [4]), implies that $\|u\|_{L^\infty(\omega)} \in O(1)$, where ω is a neighborhood of $\partial\Omega$ in Ω, through the argument by de Figueiredo-Lions-Nussbaum [2]. Hence, from elliptic boundary estimates and bootstrap argument we obtan $\|D^\alpha u\|_{L^\infty(\omega_m)} \in O(1)$ for each α with $|\alpha|=m$, where a neighborhood ω_m of $\partial\Omega$ in Ω satisfies $\omega \supset \omega_1 \supset \omega_2 \supset \cdots \supset \omega_m \supset \cdots$. On the other hand, $\Sigma = \|\Delta u\|_{L^1} \in O(1)$ implies $\|u\|_{W^{1,p}} \in O(1)$ $(p<2)$ by the L^1-estimate due to Brézis-Strauss [1]. Hence $\{u\}$ accumulates to a $v \in W^{1,p}(\Omega) \cap C^2(\omega_2)$, which is harmonic in ω_2.

Here we introduce the function $S = u_{zz} - (1/2)u_z^2$, where $z = x_1 + ix_2$ for $x = (x_1, x_2)$. Then, according to Liouville [5] we have

Proposition. *$S = S(z)$ is a holomorphic function of $z \in \Omega \subset C$. Let $\{\varphi_1, \varphi_2\}$ be the fundmental system of solutions for $\varphi_{zz} + (1/2)S(z)\varphi = 0$ satisfying $\varphi_1(z_0) = \varphi_2'(z_0) = 0$ and $\varphi_1'(z_0) = \varphi_2(z_0) = 1$, where $z_0 = x_{10} + ix_{20}$ with $x_0 = (x_{10}, x_{20}) \in \Omega$ being a maximal point of $u = u(x)$. Then we have*

$$e^{-u/2} = c^2|\varphi_1|^2 + \frac{\lambda}{8}c^{-2}|\varphi_2|^2 \tag{2.1}$$

with a real constant c.

We know that maximal points can not approach to $\partial\Omega$ by [4]. Furthermore, the holomorphic functions $\{S = S(z)\}$ are uniformly bounded near $\partial\Omega$ so that $\|S\|_{L^\infty} \in O(1)$ by the maximal principle. Hence $\|\varphi_j\|_{L^\infty_{\mathrm{loc}}(\Omega)} \in O(1)$ $(j = 1, 2)$, and $\{\varphi_j = \varphi_j(z)\}$ accumlates to some holomorphic function $\psi_j = \psi_j(z)$ in the open compact topology of $C(\Omega)$. Since $\psi_j \not\equiv 0$, the set Z_j of its zeros is discrete. Also $\{c^2\}$ accumulates to some $\rho \in [0, +\infty]$.

In the case $\rho \in (0, +\infty)$, $\{\|u\|_{L^\infty_{\mathrm{loc}}(\Omega \backslash Z_1)}\}$ is bounded so that v is C^∞ in $\bar{\Omega} \backslash Z_1$ and $e^{-v/2} = \rho|\psi_1|^2$ holds there. Furthermore $Z_1 \cap \omega = \phi$ and hence Z_1 is finite, corresponding to the blow-up set of v. In the case $\rho = +\infty$, $v \equiv 0$ follows similarly. For the case $\rho = 0$, let us suppose that $\{(\lambda/8)c^{-2}\}$ accumulates to $\mu \in [0, +\infty]$. When $\mu \in (0, +\infty)$, we similarly have that Z_2 is finite and coincides with the blow-up set of v. If $\mu = +\infty$, then $v \equiv 0$ follows. Finally, the case $\rho = \mu = 0$ contradicts to $\|u\|_{L^\infty(\omega)} \in O(1)$.

Proof of Claim 3. Since $\|S\|_{L^\infty(\Omega)} \in O(1)$, there exists a holomorphic function $T = T(z)$ in Ω such that $T(z) = v_{zz} - (1/2)v_z^2$ holds in $\Omega \backslash \{\kappa_1, \cdots, \kappa_l\}$. On the other hand $v \in W^{1,p}(\Omega)$ $(1<p<2)$ is harmonic in $\Omega \backslash \{\kappa_1\}$, which implies that $h \equiv v - \alpha E_{\kappa_1}$ is harmonic around $z = \kappa_1$, where α is a real constant. In fact, we can show that $\Delta v = C\delta(\kappa_{1_2})$ around $z = \kappa_1$ for some constant C. Now we recall $E_\kappa = -(1/4\pi)\log|G_\kappa|$, where $G = G_\kappa : \Omega \to D = \{|z|<1\}$ is a conformal mapping with $G(\kappa) = 0$.

For $w = v - h = \alpha E_{\kappa_1}$ we have $w_{zz} - (1/2)w_z^2 = -(\alpha/4\pi)\{(G''/G) + (1 - (\alpha/8\pi)) \times (G'/G)^2\}$, which is meromorphic around $z = \kappa_1$. Since $(\partial/\partial\bar{z})\{w_{zz} - (1/2)w_z^2\} =$

$-(\alpha/4)\delta(\kappa_1)h_z$ by $(\partial/\partial\bar{z})T(z)=0$, we have $\alpha=0$ or $h_z=0$. In the latter case, the function $h=h(\bar{z})=H(z)$ is anti-holomorphic and realvalued, so that is a constant. Hence $w_{zz}-(1/2)w_z^2=T(z)$ is holomorphic around $z=\kappa_1$. Thus $G''(\kappa_1)=0$ and $\alpha=8\pi$ follows because $G'\neq 0$. The former characterizes that κ_1 is a core. In other words, $z=\kappa_1$ is a removable singular point of the harmonic function $v\in C^\infty(\bar{\Omega}\backslash\{\kappa_1\})$ or is a core and $v=8\pi E_{\kappa_1}+$constant around $z=\kappa_1$. Regarding the unique continuation property of harmonic functions, we conclude that $v=8\pi E_\kappa$ because $v|_{\partial\Omega}=E_\kappa|_{\partial\Omega}=0$.

Note added in Proof. Theorem holds for general domains without simply connectedness. Solutions $\{u\}$ make finite point blow up unless they approach to 0 or make entire blow up. Blowing up points $\{\kappa_1, \cdots, \kappa_l\}$ are characterized by $k(x)=K(x, x)$ and $G(x, y)$. Details will be written in the paper cited at the end of § 1.

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