2. *Admissible Solutions of Higher Order Differential Equations*

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1. Introduction. We use here standard notations in Nevanlinna theory [3], [5].

Let $f(z)$ be a meromorphic function. As usual, $m(r,f)$, $N(r,f)$, and $T(r,f)$ denote the proximity function, the counting function, and the characteristic function of $f(z)$, respectively. Let $\bar{N}(r,f)$ be the counting function for distinct poles of $f(z)$.

A function $\varphi(r)$, $0\leq r<\infty$, is said to *be* $S(r,f)$ if there is a set $E\subset \boldsymbol{R}^{+}$ of finite linear measure such that $\varphi(r)=o(T(r,f))$ as $r\to\infty$, $r\notin E$. A meromorphic function $a(z)$ is said to be *small with respect to* $f(z)$ if $T(r,a)=S(r,f)$. Let $a_j(z)$, $j=1,\cdots,n$, be meromorphic functions. A function $w(z)$ is *admissible with respect to* $a_j(z)$, if $T(r,a_j)=S(r,w)$, $j=1,\cdots,n$.

For a differential monomial $M[w]=a(z)w^{n_0}(w')^{n_1}\cdots(w^{(m)})^{n_m}$ in w, we put $\gamma_M=n_0+n_1+\cdots+n_m$ and $\Gamma_M^{\mu}=\mu n_0+(\mu+1)n_1+\cdots+(\mu+m)n_m$, and call *degree* and *weight-*μ of $M[w]$, respectively. We write Γ_M^1 simply as Γ_M. Let $\Omega(z)$ be a differential polynomial with meromorphic coefficients:

$$\Omega[w]=\sum_{\lambda\in I}M_\lambda[w]=\sum_{\lambda\in I}a_\lambda(z)w^{n_0}(w')^{n_1}\cdots(w^{(m)})^{n_m},$$

where $a_\lambda(z)$ are meromorphic functions, I is a finite set of multi-indices $\lambda=(n_0,n_1,\cdots,n_m)$. We define *degree* γ_Ω and *weight-*μ Γ_Ω^{μ} of Ω by $\gamma_\Omega=\max_{\lambda\in I}\gamma_{M_\lambda}$ and $\Gamma_\Omega^{\mu}=\max_{\lambda\in I}\Gamma_{M_\lambda}^{\mu}$, respectively.

A meromorphic solution $w(z)$ of the differential equation $\Omega[w]=0$ is *admissible solution*, if $w(z)$ is admissible w.r.t. $a_\lambda(z)$, $\lambda\in I$.

$\Omega[w]$ is said to *satisfy the condition* (GL) *if, for any* $\mu\geq 1$,

(GL) *there is an index* i_μ *such that* $\Gamma_{M_{i_\mu}}^{\mu}>\Gamma_{M_i}^{\mu}$ *if* $i\neq i_\mu$.

This condition (GL) is due to Gackstatter-Laine [2], who investigated the equation

$$w'^n=\sum_{j=0}^{m}a_j(z)w^j \qquad (0\leq m\leq 2n), \tag{1.1}$$

and conjectured that it would not admit any admissible solution if $1\leq m\leq n-1$. In this respect, Toda [7] proved the following theorem.

Theorem A. *The differential equation* (1.1) *does not possess any admissible solutions if* $1\leq m\leq n-1$, *except for the case when* $n-m$ *is a divisor of* n *and* (1.1) *is of the following form*:

$$w'^n=a_m(z)(w+\alpha)^m, \qquad \text{where } \alpha \text{ is a constant.}$$

Recently, Toda [8] studied more general differential equation

(1.2) $$P[w]^n=\sum_{j=0}^{m} a_j(z)w^j$$

with a differential polynomial $P[w]$ instead of w'. He proved

Theorem B. *If $1\leq m\leq n-1$, the equation (1.2) does not possess any admissible solutions except for the case when it is of the form*

$$P[w]^n=a_m(z)(w+b(z))^m.$$

In connection with the conjecture of Gackstatter-Laine and theorems of Toda, we will study the following: Let $H[w]$ and $F[w]$ be differential polynomials with meromorphic coefficients. Suppose the equation

(1.3) $$H[w]^n=F[w]$$

possesses an admissible solution. Find smallest integer n_0 such that, if $n\geq n_0$, then the form of $F[w]$ is decided.

By Theorems A and B, we see that $n_0=m+1$ if $F[w]$ is a (not differential) polynomial of degree m.

In this note, we will prove the following result.

Theorem 1. *Let $H[w]$, $P[w]$ and $Q[w]$ be differential polynomials with meromorphic coefficients. Suppose $H[w]$ and $P[w]$ are not identically zero, $H[w]$ satisfies the condition (GL), and the equation*

(1.4) $$H[w]^n=w^m P[w]+Q[w]$$

admits an admissible solution. If $n>\max\{(\Gamma_P+m)/\gamma_H, (\gamma_P+1)(\gamma_P+m)\}$ and $m-2\geq\Gamma_Q$, then we have $Q[w]\equiv 0$.

2. Preliminary lemmas. We use the following notation $\omega(z_0, g)$. Let $g(z)$ be meromorphic. $\omega(z_0, g)=m$ if $g(z)$ has a pole of order m at z_0; $\omega(z_0, g)=0$ if $g(z_0)\neq\infty$.

In the sequel, $w=w(z)$ denotes a meromorphic function. Differential polynomial of w with meromorphic coefficients is called simply as d.p.m.c. of w.

Lemma 1 ([1], [4]). *Let $P[w]$ be a d.p.m.c. of w, then*

$$m(r, P[w])\leq\gamma_P m(r, w)+S(r, w).$$

Lemma 2 ([4]). *Let $\Psi[w]$ be a d.p.m.c. of w with the form*

$$\Psi[w]=w^m P[w]+Q[w],$$

where $P[w]$ and $Q[w]$ are d.p.m.c. Suppose $Q[w]\not\equiv 0$ and $\Gamma_Q\leq n-2$. Then

$$T(r, w)\leq\overline{N}(r, w)+(\gamma_P+1)\overline{N}(r, 1/\Psi)+S(r, w).$$

Lemma 3 ([3], [6]). *Let $Q[w]$ and $Q^*[w]$ be d.p.m.c. of w with coefficients a_j and a_k^*, respectively, and $G(w)$ be a polynomial of w with constant coefficients. Suppose that $m(r, a_j)$ and $m(r, a_k^*)$ are $S(r, w)$. If $\gamma_Q\leq\gamma_G$ and $Q[w]=G(w)Q^*[w]$, then $m(r, Q^*[w])=S(r, w)$.*

Lemma 4. *Let $Q[w]$ and $P[w]$ be d.p.m.c. of w and $G(w)$ be a polynomial of w with constant coefficients. Suppose $P[w]\not\equiv 0$. If the equation*

(2.1) $$Q[w]=G(w)P[w]$$

possesses an admissible solution $w(z)$, then we have

(2.2) $$\gamma_G(\text{degree of } G)\leq\Gamma_Q.$$

Proof. Suppose $\gamma_G>\Gamma_Q$. Let z_0 be a pole of w which is neither zero nor pole for coefficients of $P[w]$ and $Q[w]$. Put $\omega(z, w)=\mu\geq 1$ and

$\omega(z_0, P[w])=\nu$, then $\omega(z_0, G(w)P[w])=\mu\gamma_G+\nu$. By (2.1) and $\omega(z_0, Q[w])\leq\mu\Gamma_Q$, we get $\mu\gamma_G+\nu\leq\mu\Gamma_Q$, i.e., $\nu\leq\mu(\Gamma_Q-\gamma_G)<0$, which is a contradiction. Hence such a pole z_0 of w does not exist, which implies $N(r, w)=S(r, w)$, since w is admissible. Therefore

(2.3) $$N(r, P[w])\leq\Gamma_P N(r, w)+S(r, w)=S(r, w).$$

By the assumption $\gamma_G>\Gamma_Q\geq\gamma_Q$. By Lemma 3 we get

(2.4) $$T(r, P[w])=S(r, w).$$

By (2.3) and (2.4),

(2.5) $$T(r, P[w])=S(r, w).$$

By (2.1), (2.5), and Lemma 1,

$$\begin{aligned}\gamma_G T(r, w)+S(r, w)&=T(r, G(w)P[w])=T(r, Q[w])\\&=m(r, Q[w])+N(r, Q[w])\leq\gamma_Q m(r, w)+\Gamma_Q N(r, w)+S(r, w)\\&\leq\Gamma_Q T(r, w)+S(r, w),\end{aligned}$$

and hence $(\gamma_G-\Gamma_Q)T(r, w)\leq S(r, w)$, a contradiction. Thus $\gamma_G\leq\Gamma_Q$.

3. Proof of Theorem 1. Suppose $Q[w]\not\equiv 0$. Put

(3.1) $$\Psi[w]=w^m P[w]+Q[w].$$

Since $m>\Gamma_Q$, the admissible solution $w(z)$ of (1.5) does not satisfy the equation $w^m P[w]+Q[w]=0$ by Lemma 4.

Let z_0 be a pole of w which is neither zero nor pole for coefficients of $H[w]$, $P[w]$ and $Q[w]$. Put $\omega(z_0, w)=\mu$. By the condition (GL) for $H[w]$ and by the assumption in the theorem

$$n\mu\gamma_H\leq\omega(z_0, H[w]^n)=\omega(z_0, \Psi)\leq\mu(m+\Gamma_P)<n\mu\gamma_H,$$

which is a contradiction. Hence there is not such a z_0, hence

(3.2) $$N(r, w)=S(r, w).$$

We note that $T(r, H[w])=O(T(r, w))$ and

(3.3) $$T(r, \Psi)=T(r, H[w]^n)=nT(r, H[w])+S(r, H[w]).$$

We obtain by Lemma 1 and (3.2)

(3.4) $$\begin{aligned}T(r, \Psi)&=m(r, \Psi)+N(r, \Psi)\leq(\gamma_P+m)m(r, w)+\Gamma_P N(r, w)\\&+S(r, w)\leq(\gamma_P+m)T(r, w)+S(r, w).\end{aligned}$$

By Lemma 2 and (3.2)

(3.5) $$\begin{aligned}T(r, w)&\leq\overline{N}(r, w)+(\gamma_P+1)\overline{N}\left(r, \frac{1}{\Psi}\right)+S(r, w)\\&\leq(\gamma_P+1)\overline{N}\left(r, \frac{1}{H}\right)+S(r, w)\leq(\gamma_P+1)T(r, H)+S(r, w).\end{aligned}$$

From (3.3), (3.4) and (3.5)

$$\begin{aligned}T(r, w)&\leq[(\gamma_P+1)/n]T(r, \Psi)+S(r, w)\\&\leq[(\gamma_P+1)(\gamma_P+m)/n]T(r, w)+S(r, w),\end{aligned}$$

hence $\{1-[(\gamma_P+1)(\gamma_P+m)/n]\}T(r, w)\leq S(r, w)$, which is a contradiction. Thus our theorem is proved.

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