73. *Theta Series and the Poincaré Divisor*

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Let H_n be the Siegel upperhalf space of degree n, that is, $H_n = \{z \in M_n(C) \mid {}^t z = z,\ \mathcal{I}m\, z > 0\}$. Then the classical theta $\vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}(z \mid x)$ may be regarded as a function of (z, k', k'', x) on $H_n \times R^n \times R^n \times C^n$. Now we introduce a complex variable $k = zk' + k''$, and after a minor modification of $\vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}(z \mid x)$, we define a new series $\vartheta(z, k, x)$, which represents a holomorphic function on the space $H_n \times C^n \times C^n$ whose second factor C^n will be regarded as the dual space of the third factor C^n in a natural way. This new function $\vartheta(z, k, x)$ substitutes for the classical theta and sometimes has an advantage because of its complex analyticity. For instance, using this function we can explicitly write down a theta function whose divisor is the Poincaré divisor.

1. The dual lattice. Let (E, G) be a pair of n-dimensional C-vector space E and a lattice subgroup G. Assume that the quotient E/G is an abelian variety, or equivalently that there are a C-basis $(\mathfrak{e}_1, \cdots, \mathfrak{e}_n)$ and an R-basis $(\mathfrak{f}_1, \cdots, \mathfrak{f}_{2n})$ of E such that $(\mathfrak{f}_1, \cdots, \mathfrak{f}_{2n}) = (\mathfrak{e}_1, \cdots, \mathfrak{e}_n)(z\ \ 1_n)$ with a matrix z in the Siegel upperhalf space H_n and the identity n-matrix 1_n (which is sometimes denoted simply by 1), and that G is generated by $(\mathfrak{e}_1, \cdots, \mathfrak{e}_n)(z\ \ e)$ with an $(n \times n)$-matrix e having Z-coefficients and $\det e \neq 0$. Under this C-basis, E is identified with C^n and G is generated by the column vectors of $(z\ \ e)$, denoted by $G = \langle z\ \ e \rangle$. The R-coordinates $\boldsymbol{x} = \begin{pmatrix} x' \\ x'' \end{pmatrix}$, x' and $x'' \in R^n$, of a point $x \in C^n$ under the latter basis are determined by $x = (z\ \ 1_n)\boldsymbol{x} = zx' + x''$.

The classical theta series $\vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}(z \mid x)$ is defined by

$$\vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}(z \mid x) = \sum_{r \in Z^n} \boldsymbol{e}\left(\frac{1}{2}{}^t(r+k')z(r+k') + {}^t(r+k')(x+k'')\right),$$

where (z, k', k'', x) are variables on $H_n \times R^n \times R^n \times C^n$, and for each $s = (z\ \ 1)\begin{pmatrix} s' \\ s'' \end{pmatrix}$, $s', s'' \in Z^n$, we have

$$\vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}(z \mid x+s) = \vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}(z \mid x)\boldsymbol{e}\left(-{}^t s' x - \frac{1}{2}{}^t s' z s' - {}^t k'' s' + {}^t k' s''\right),$$

which suggests that $\begin{pmatrix} -k'' \\ k' \end{pmatrix}$ should be regarded as the R-coordinates of a point $\mathfrak{k}$ of the dual space $\hat{E} = \mathrm{Hom}_R(E, C)/\mathrm{Hom}_C(E, C)$ of $E = C^n$, which is naturally identified with $\mathrm{Hom}_R(E, R)$ by the restriction of the projection map π: $\mathrm{Hom}_{\boldsymbol{R}}(E, \boldsymbol{C}) \rightarrow \hat{E}$. On the other hand the space $\hat{E}$ is also isomorphic to the space $\overline{\mathrm{Hom}}_{\boldsymbol{C}}(E, \boldsymbol{C})$ of anti-linear forms on E by $2\sqrt{-1}$ times the projection, and accordingly has a structure of n-dimensional $\boldsymbol{C}$-vector space.

These two identifications of $\hat{E}$ with $\overline{\mathrm{Hom}}_{\boldsymbol{C}}(E, \boldsymbol{C})$ and with $\mathrm{Hom}_{\boldsymbol{R}}(E, \boldsymbol{R})$ give rise to the two bilinear forms on $E \times \hat{E}$,

$$\text{a sesquilinear one } [\cdot, \cdot]: \quad E \times \hat{E} \longrightarrow \boldsymbol{C},$$

and

$$\text{an } \boldsymbol{R}\text{-bilinear one } I(\cdot, \cdot): \quad E \times \hat{E} \longrightarrow \boldsymbol{R}$$

satisfying $I(x, k) = \mathcal{I}m\,[x, k]$ for $(x, k) \in E \times \hat{E}$.

Now, let $(\hat{\mathfrak{e}}_1, \cdots, \hat{\mathfrak{e}}_n)$ be the $\boldsymbol{C}$-basis of $\hat{E}$ dual to $(\mathfrak{e}_1, \cdots, \mathfrak{e}_n)$ with respect to $[\cdot, \cdot]$, and $(\hat{\mathfrak{f}}_1, \cdots, \hat{\mathfrak{f}}_{2n})$ the $\boldsymbol{R}$-basis dual to $(\mathfrak{f}_1, \cdots, \mathfrak{f}_{2n})$ with respect to $I(\cdot, \cdot)$. Then we have

$$(\hat{\mathfrak{f}}_1, \cdots, \hat{\mathfrak{f}}_{2n}) = (\hat{\mathfrak{e}}_1, \cdots, \hat{\mathfrak{e}}_n)(\mathcal{I}m\,z)^{-1}(-1 \quad z).$$

We take $(\hat{\mathfrak{f}}_{n+1}, \cdots, \hat{\mathfrak{f}}_{2n}, -\hat{\mathfrak{f}}_1, \cdots, -\hat{\mathfrak{f}}_n)$ and $(\hat{\mathfrak{e}}_1, \cdots, \hat{\mathfrak{e}}_n)(\mathcal{I}m\,z)^{-1}$ as $\boldsymbol{R}$- and $\boldsymbol{C}$-coordinate vectors on $\hat{E}$, respectively, and write, for $\mathfrak{k} \in \hat{E}$

$$\mathfrak{k} = (\hat{\mathfrak{f}}_{n+1}, \cdots, \hat{\mathfrak{f}}_{2n}, -\hat{\mathfrak{f}}_1, \cdots, -\hat{\mathfrak{f}}_n)\begin{pmatrix} k' \\ k'' \end{pmatrix} = (\hat{\mathfrak{e}}_1, \cdots, \hat{\mathfrak{e}}_n)(\mathcal{I}m\,z)^{-1}k,$$

where $k \in \boldsymbol{C}^n$, k' and $k'' \in \boldsymbol{R}^n$ with $k = (z \quad 1)\begin{pmatrix} k' \\ k'' \end{pmatrix}$. The space $\hat{E}$ is identified with $\boldsymbol{C}^n$ under this $\boldsymbol{C}$-coordinate system. Using these notation we have

$$[x, k] = {}^t\bar{x}(\mathcal{I}m\,z)^{-1}k,$$

and

$$I(x, k) = -{}^t x' k'' + {}^t x'' k'.$$

(1.1) Let $\boldsymbol{G} = \langle z \quad e \rangle$ be a lattice subgroup of E. Then the lattice subgroup $\hat{\boldsymbol{G}}$ of $\hat{E}$ dual to $\boldsymbol{G}$ is defined by

$$\begin{aligned} \hat{\boldsymbol{G}} &= \{k \in \hat{E} \mid I(s, k) \in \boldsymbol{Z} \text{ for any } s \in \boldsymbol{G}\} \\ &= \langle z\,{}^t e^{-1} \quad 1 \rangle = \{zq' + q'' \mid q' \in {}^t e^{-1}\boldsymbol{Z}^n,\ q'' \in \boldsymbol{Z}^n\}. \end{aligned}$$

2. The function $\vartheta(z, k, x)$.

Definition (2.0). A holomorphic function $\vartheta(z, k, x)$ on $H_n \times \hat{E} \times E$ is defined by

$$\begin{aligned} \vartheta(z, k, x) &= \sum_{r \in \boldsymbol{Z}^n} \boldsymbol{e}\left(\frac{1}{2}{}^t(r + z^{-1}k + z^{-1}x)z(r + z^{-1}k + z^{-1}x)\right) \\ &= \boldsymbol{e}\left(\frac{1}{2}{}^t(x + k'')z^{-1}(x + k'')\right)\vartheta\begin{bmatrix} k' \\ k'' \end{bmatrix}(z \mid x) \\ &= \boldsymbol{e}\left(\frac{1}{2}{}^t(k + x)z^{-1}(k + x)\right)\vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(z \mid k + x). \end{aligned}$$

This function actually depends only on $(z, k + x)$.

(2.1) For a fixed z, the function $\vartheta(z, k, x)$ satisfies the relation: for $q = (z \quad 1)\begin{pmatrix} q' \\ q'' \end{pmatrix} = (z\,{}^t e^{-1} \quad 1)\begin{pmatrix} q'_0 \\ q''_0 \end{pmatrix}$, $q' = {}^t e^{-1} q'_0 \in {}^t e^{-1}\boldsymbol{Z}^n$ (or, $q'_0 \in \boldsymbol{Z}^n$), $q'' = q''_0 \in \boldsymbol{Z}^n$, and $s = (z \quad 1)\begin{pmatrix} s' \\ s'' \end{pmatrix} = (z \quad e)\begin{pmatrix} s'_0 \\ s''_0 \end{pmatrix}$, $s' = s'_0 \in \boldsymbol{Z}^n$, $s'' = e s''_0 \in e\boldsymbol{Z}^n$ (or, $s''_0 \in \boldsymbol{Z}^n$)

$$\begin{aligned} &\vartheta(z, k + q, x + s) \\ &\quad = \vartheta(z, k + zq', x)\,\boldsymbol{e}\left({}^t(q'' + s'')z^{-1}(k + x) + \frac{1}{2}{}^t(q'' + s'')z^{-1}(q'' + s'') + {}^t q' q''\right). \end{aligned}$$

Proof. In fact,

$$\vartheta(z, k+q, x+s)=\sum_{r\in \boldsymbol{Z}^n} \boldsymbol{e}\Big(\frac{1}{2}{}^t(r+z^{-1}k+q'+z^{-1}q''+z^{-1}x+s'+z^{-1}s'')$$
$$\times z(r+z^{-1}k+q'+z^{-1}q''+z^{-1}x+s'+z^{-1}s'')\Big),$$

(applying the substituion of $r+s'$ by r, and the congruence equality ${}^ts''q'\equiv 0 \bmod \boldsymbol{Z}$,)

$$=\sum_{r\in \boldsymbol{Z}^n} \boldsymbol{e}\Big(\frac{1}{2}{}^t(r+z^{-1}(k+zq')+z^{-1}x)z(r+z^{-1}(k+zq')+z^{-1}x)$$
$$+{}^t(q''+s'')(r+z^{-1}k+q'+z^{-1}x)+\frac{1}{2}{}^t(q''+s'')z^{-1}(q''+s'')$$
$$=\vartheta(z, k+zq', x)\boldsymbol{e}\Big({}^t(q''+s'')z^{-1}(k+x)+\frac{1}{2}{}^t(q''+s'')z^{-1}(q''+s'')+{}^tq'q''\Big). \qquad \square$$

(2.2) The function $\vartheta(z, k, x)$ of $\begin{pmatrix} k \\ x \end{pmatrix}$ is periodic with period $\begin{pmatrix} z\boldsymbol{Z}^n \\ z\boldsymbol{Z}^n \end{pmatrix}$, and a theta function with respect to the period matrix $\begin{pmatrix} z & 1_n & 0 & 0 \\ 0 & 0 & z & 1_n \end{pmatrix}$.

For $q=zq'+q''$ and $s=zs'+s''$ with $q', q'', s', s''\in \boldsymbol{Z}^n$, we have

$$\vartheta(z, k+q, x+s)=\vartheta(z, k, x)\boldsymbol{e}\Big({}^t(q''+s'')z^{-1}(k+x)+\frac{1}{2}{}^t(q''+s'')z^{-1}(q''+s'')\Big),$$
$$\vartheta(z, k, x+s)=\vartheta(z, k, x)\boldsymbol{e}\Big({}^ts''z^{-1}x+\frac{1}{2}{}^ts''z^{-1}s''+{}^tkz^{-1}s''\Big)$$
$$=\vartheta(z, k, x)\boldsymbol{e}\Big({}^ts''z^{-1}x+\frac{1}{2}{}^ts''z^{-1}s''+I(s, k)+{}^tk''z^{-1}s\Big)$$

and the similar formula for $\vartheta(z, k+q, x)$.

According to these formulas we know that for fixed z, k_1, k_2, the following three conditions for the two theta functions $\vartheta(z, k_i, x)$, $i=1, 2$, of x with respect to $\langle z\ 1\rangle$ are equivalent:

(a) $k_1\equiv k_2 \bmod \langle z\ 1\rangle$.

(b) The two theta functions $\vartheta(z, k_1, x)$ and $\vartheta(z, k_2, x)$ coincide up to a trivial theta function factor.

(c) The two theta functions $\vartheta(z, k_1, x)$ and $\vartheta(z, k_2, x)$ are of the same type up to a factor of a trivial theta function.

The statement obtained by exchanging x and k in the above is obviously true, and we get

Proposition (2.3). *The theta function $\vartheta(z, k, x)$ determines the* Poincaré divisor *on the product of the abelian variety $\boldsymbol{C}^n/\langle z\ 1\rangle$ and its dual $\boldsymbol{C}^n/\langle z\ 1\rangle$.*

3. The function $\eta_e(z, k, x)$.

Notation. Let e be a matrix as above, let ε be the smallest positive integer such that $\varepsilon e^{-1}\in \boldsymbol{M}(n, \boldsymbol{Z})$, and let $U({}^te)$ be a complete set of representatives of ${}^te^{-1}\boldsymbol{Z}^n \bmod \boldsymbol{Z}^n$.

Definition (3.0). A holomorphic function $\eta_e(z, k, x)$ on $H_n\times \boldsymbol{C}^n\times \boldsymbol{C}^n$ is defined by

$$\eta_e(z, k, x)=\sum_{p'\in U({}^te)} \vartheta(z, k+zp', x)(\vartheta(z, k+zp', 0))^{\varepsilon-1}.$$

The function $\eta_e(z, k, x)$ is well-defined, that is, independent of the choice of the representatives $U({}^te)$. (See (2.2).)

(3.1) Let q and s be as in (2.1). Then

$$\eta_e(z, k+q, x+s)$$
$$=\eta_e(z, k, x)\boldsymbol{e}\Big({}^t(\varepsilon q''+s'')z^{-1}k+{}^t(q''+s'')z^{-1}x+\frac{\varepsilon}{2}{}^tq''z^{-1}q''+{}^tq''z^{-1}s''+\frac{1}{2}{}^ts''z^{-1}s''\Big)$$
$$=\eta_e(z, k, x)\boldsymbol{e}\left({}^t\begin{pmatrix} q'_0 \\ q''_0 \\ s'_0 \\ s''_0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \varepsilon z^{-1} & z^{-1} \\ 0 & 0 \\ {}^tez^{-1} & {}^tez^{-1} \end{pmatrix}\begin{pmatrix} k \\ x \end{pmatrix}+{}^t\begin{pmatrix} q'_0 \\ q''_0 \\ s'_0 \\ s''_0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\varepsilon z^{-1} & 0 & \frac{1}{2}z^{-1}e \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}{}^tez^{-1} & 0 & \frac{1}{2}{}^tez^{-1}e \end{pmatrix}\begin{pmatrix} q'_0 \\ q''_0 \\ s'_0 \\ s''_0 \end{pmatrix}\right).$$

Thus the function $\eta_e(z, k, x)$ is a theta function of variables (k, x) of type $\left(\begin{pmatrix} z{}^te^{-1} & 1 & 0 & 0 \\ 0 & 0 & z & e \end{pmatrix}\Big| h, (B, \boldsymbol{m}), f, l\right)$, that is,

the theta factor part in the above

$$=\boldsymbol{e}\left(\frac{1}{2\sqrt{-1}}\left({}^t\begin{pmatrix} \bar{q} \\ \bar{s} \end{pmatrix}h+{}^t\begin{pmatrix} q \\ s \end{pmatrix}f\right)\begin{pmatrix} k \\ x \end{pmatrix}+\frac{1}{4\sqrt{-1}}\left({}^t\begin{pmatrix} \bar{q} \\ \bar{s} \end{pmatrix}h+{}^t\begin{pmatrix} q \\ s \end{pmatrix}f\right)\begin{pmatrix} q \\ s \end{pmatrix}\right.$$
$$\left.+\frac{1}{2}{}^t\begin{pmatrix} q'_0 \\ q''_0 \\ s'_0 \\ s''_0 \end{pmatrix}B\begin{pmatrix} q'_0 \\ q''_0 \\ s'_0 \\ s''_0 \end{pmatrix}+{}^t\boldsymbol{m}\begin{pmatrix} q'_0 \\ q''_0 \\ s'_0 \\ s''_0 \end{pmatrix}+{}^tl\begin{pmatrix} q \\ s \end{pmatrix}\right),$$

where

$$h=\begin{pmatrix} \varepsilon(\mathcal{J}m\, z)^{-1} & (\mathcal{J}m\, z)^{-1} \\ (\mathcal{J}m\, z)^{-1} & (\mathcal{J}m\, z)^{-1} \end{pmatrix}, \qquad f=-h+2\sqrt{-1}\begin{pmatrix} \varepsilon z^{-1} & z^{-1} \\ z^{-1} & z^{-1} \end{pmatrix},$$
$$B=\begin{pmatrix} 0 & 0 & 0 & 0 \\ \varepsilon{}^te^{-1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & {}^te & 0 \end{pmatrix}, \qquad \boldsymbol{m}=0 \quad \text{and} \quad l=0.$$

Proof. Applying the formula (2.1) to our case we have the first equality, and can directly check the other parts.

(3.2) If we put $q'=q''=0$ in the above equality, then for $s=(z\ \ 1)\begin{pmatrix} s' \\ s'' \end{pmatrix}\in \boldsymbol{G}$, $s'\in \boldsymbol{Z}^n$, $s''\in e\boldsymbol{Z}^n$,

$$\eta_e(z, k, x+s)=\eta_e(z, k, x)\boldsymbol{e}\Big(\frac{1}{2}{}^ts''z^{-1}s''+{}^t(k+x)z^{-1}s''\Big)$$
$$=\eta_e(z, k, x)\boldsymbol{e}\Big({}^ts''z^{-1}x+\frac{1}{2}{}^ts''z^{-1}s''+I(s, k)+{}^tk''z^{-1}s\Big).$$

In the same way, for $q=(z\ \ 1)\begin{pmatrix} q' \\ q'' \end{pmatrix}\in \hat{\boldsymbol{G}}$, $q'\in {}^te^{-1}\boldsymbol{Z}^n$, $q''\in \boldsymbol{Z}^n$,

$$\eta_e(z, k+q, x)=\eta_e(z, k, x)\boldsymbol{e}\Big(\frac{1}{2}\varepsilon{}^tq''z^{-1}q''+{}^t(\varepsilon k+x)z^{-1}q''\Big)$$
$$=\eta_e(z, k, x)\boldsymbol{e}\Big(\varepsilon{}^tq''z^{-1}k+\frac{1}{2}\varepsilon{}^tq''z^{-1}q''-I(x, q)+{}^tx''z^{-1}q\Big).$$

Proof. These are special cases of the formula (3.1) and we have only to remark

$${}^tkz^{-1}s''=-{}^tk''s'+{}^tk's''+{}^tk''z^{-1}s=I(s, k)+{}^tk''z^{-1}s,$$

and

$$ {}^{t}xz^{-1}q'' = -{}^{t}x''q' + {}^{t}x'q'' + {}^{t}x''z^{-1}q = -I(x, q) + {}^{t}x''z^{-1}q. \qquad \square $$

For a fixed (z, k), the function $\eta_e(z, k, x)$ of x is a theta function with respect to $\langle z\ \ e\rangle$. The first formula in (3.2) says that if two theta functions $\eta_e(z, k_1, x)$ and $\eta_e(z, k_2, x)$ are of the same type up to a factor of a trivial theta function, then $k_1 - k_2 \in \langle z{}^{t}e^{-1}\ \ 1_n\rangle$, and the second formula says that if $k_1 - k_2 \in \langle z{}^{t}e^{-1}\ \ 1_n\rangle$, then $\eta_e(z, k_2, x)$ is the product of $\eta_e(z, k_1, x)$ and a certain trivial theta. The similar statement for the function $\eta_e(z, k, x)$ of k naturally holds true. Thus, we have

Theorem (3.3). *For fixed z and e, the holomorphic function $\eta_e(z, k, x)$ of (k, x) on $C^n \times C^n$ is a theta function with respect to the lattice group $\langle z{}^{t}e^{-1}\ \ 1\rangle \times \langle z\ \ e\rangle$. The two abelian varieties $C^n/\langle z\ \ e\rangle$ and $C^n/\langle z{}^{t}e^{-1}\ \ -1\rangle$ are dual to each other, and the divisor X of $\eta_e(z, k, x)$ on the product of the two varieties is a corresponding* Poincaré divisor.

Note (3.4). If we are just interested in theta functions with the Poincaré divisor on $C^n/\langle z{}^{t}e^{-1}\ \ -1\rangle \times C^n/\langle z\ \ e\rangle$, it is easily seen that the functions $\vartheta\begin{bmatrix}0\\0\end{bmatrix}(z\,|\,k+x)$ and

$$ \xi_e(z, k, x) = \sum_{p' \in U({}^{t}e)} \vartheta\begin{bmatrix}p'\\0\end{bmatrix}(z\,|\,k+x)\left(\vartheta\begin{bmatrix}p'\\0\end{bmatrix}(z\,|\,k)\right)^{\varepsilon-1} $$

stand, respectively, for the above $\vartheta(z, k, x)$ and $\eta_e(z, k, x)$.

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