72. Iwasawa's λ-invariants of Certain Real Quadratic Fields

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We studied Greenberg's conjecture (cf. [3]) on real quadratic case in previous papers [1] and [2]. Two natural numbers n_1 and n_2 were defined in [1]. We treated the case $n_1<n_2$ in [1] and the case $n_1=n_2=2$ in [2]. In this paper, we shall make further investigation in the case $n_1=n_2=2$.

Let k be a real quadratic field with class number h, p an odd prime number which splits in $k/\boldsymbol{Q}$ and

$$k=k_0\subset k_1\subset\cdots\subset k_n\subset\cdots\subset k_\infty$$

the cyclotomic $\boldsymbol{Z}_p$-extension with Galois group $G(k_\infty/k)=\overline{\langle\sigma\rangle}$. Let $p=\mathfrak{p}\mathfrak{p}'$ be the prime factorization of p in k and $\mathfrak{p}_n$ (resp. $\mathfrak{p}'_n$) the unique prime ideal of k_n lying above $\mathfrak{p}$ (resp. $\mathfrak{p}'$). Let A_n be the p-primary part of the ideal class group of k_n and put $D_n=\langle\mathrm{cl}(\mathfrak{p}_n)\rangle\cap A_n$, $B_n^{(r)}=\{a\in A_n\,|\,a^{\sigma_r-1}=1\}$ for $0\leqq r\leqq n$ where $\sigma_r=\sigma^{p^r}$. We put $B_n=B_n^{(0)}$. The norm maps $N_{n,m}: k_n\to k_m$ $(0\leqq m\leqq n)$ are applied to A_n, the unit group E_n of k_n and etc.

From now on we assume that $n_1=n_2=2$. (See [1] on the definition of n_1 and n_2.) In this case, the following lemma which was proved in [1] and [3] is fundamental.

Lemma 1. *Let k be a real quadratic field and p an odd prime number which splits in $k/\boldsymbol{Q}$. Assume that*

(1) $n_1=n_2=2$, *and*

(2) $A_0=1$.

Then, $|B_n|=p$, $E_0\cap N_{n,0}(k_n^\times)=E_0^{p^{n-1}}$, *and* $(B_n: D_n)=(E_0\cap N_{n,0}(k_n^\times): N_{n,0}(E_n))$ *for all* $n\geqq 1$. *Futhermore,* $\mu_p(k)=\lambda_p(k)=0$ *if and only if* $D_n\neq 1$ *for some* $n\geqq 1$.

Now we assume that $D_r=1$ for some $r\geqq 1$ and choose $\alpha_r\in k_r$ such that $\mathfrak{p}_r'^h=(\alpha_r)$. We define the natural number $n_1^{(r)}$ by

$$\mathfrak{p}^{n_1^{(r)}}\,\|\,(N_{r,0}(\alpha_r)^{p-1}-1).$$

Since $N_{r,0}(E_r)=E_0^{p^r}$ from Lemma 1, $n_1^{(r)}$ is uniquely determined under the condition $r+1\leqq n_1^{(r)}\leqq r+2$. For $k^*=k(e^{2\pi\sqrt{-1}/p})$, we have the following result.

Proposition. *Let k and p be as in Lemma* 1. *In addition to the assumptions* (1) *and* (2) *of Lemma* 1, *we assume that*

(3) $\lambda_p^-(k^*)=1$, *and*

(4) $D_r=1$ *for some* $r\geqq 1$.

Then, $D_{r+1}\neq 1$ *is and only if* $n_1^{(r)}=r+1$. *In particular,* $\mu_p(k)=\lambda_p(k)=0$ *if* $n_1^{(r)}=r+1$.

For the Proof of Proposition, we need some lemmas. Let K_n denote the completion of k_n at $\mathfrak{p}_n$. Let $U_n=\{u\in K_n: \text{unit}\,|\,u\equiv 1 \pmod{\mathfrak{p}_n}\}$ and $U_n^{(r)}=\{u\in U_n\,|\,N_{n,0}(u)\equiv 1 \pmod{p^{n+r+1}}\}$ for $0\leqq r\leqq n$.

Lemma 2. *Under the same assumptions as in Lemma 1, $N_{n+1,n}(U_{n+1})=U_n^{(1)}$ for all $n\geqq 0$.*

Proof. Clearly $N_{n+1,n}(U_{n+1})\subset U_n^{(1)}\subset U_n$. The composite map of $N_{n,0}$: $U_n\to 1+p^{n+1}Z_p$ and $1+p^{n+1}Z_p\to 1+p^{n+1}Z_p/1+p^{n+2}Z_p$ is surjective and its kernel is $U_n^{(1)}$. Therefore $U_n/U_n^{(1)}\cong Z/pZ$. On the other hand, we see that $U_n/N_{n+1,n}(U_{n+1})\cong G(K_{n+1}/K_n)\cong Z/pZ$ by local class field theory. Hence $N_{n+1,n}(U_{n+1})=U_n^{(1)}$.

Lemma 3. *Assume that A_n is cyclic in addition to the assumptions of Lemma 1. If $D_n=1$ for some $n\geqq 1$, then $A_{n+1}=B_{n+1}^{(n)}$ and its order is p^{n+1}.*

Proof. We proceed by induction on n. First we have to show that $A_1=B_1$. Note that $|B_1|=p$ from Lemma 1. Assume that $B_1\subsetneqq A_1$. Then there exists $a\in A_1$ such that $a^{\sigma-1}\neq 1$ and $a^{(\sigma-1)^2}=1$. It is easy to see that there exist $u\in Z_p[G(k_1/k)]^\times$ and $v\in Z_p[G(k_1/k)]$ such that $1+\sigma+\cdots+\sigma^{p-1}=(\sigma-1)^2v+pu$. Since $|A_0|=1$, we see that $a^p=1$ and $a\in B_1$ because A_1 is cyclic by assumption, and this is a contradiction. Next we assume that proposition holds for $n-1$. Since $D_n=1$, $N_{n,0}(E_n)=E_0^{p^n}$ from Lemma 1. It follows from Lemma 2 that an element of E_n is a local norm from k_{n+1} at $\mathfrak{p}_n$. Since any place which does not lie above p is unramified in k_{n+1}/k_n, the product formula of norm residue symbol and Hasse's norm theorem imply that $E_n\subset N_{n+1,n}(k_{n+1}^\times)$. Then by the genus theory for k_{n+1}/k_n,

$$|B_{n+1}^{(n)}|=|A_n|\frac{p^2}{p(E_n: E_n\cap N_{n+1,n}(k_{n+1}^\times))}=p^{n+1}.$$

Now assume that $B_{n+1}^{(n)}\subsetneqq A_{n+1}$ and choose $a\in A_{n+1}$ such that $a^{\sigma_n-1}\neq 1$ and $a^{(\sigma_n-1)^2}=1$. As above, by taking $u\in Z_p[G(k_{n+1}/k_n)]^\times$ and $v\in Z_p[G(k_{n+1}/k_n)]$ such that $1+\sigma_n+\cdots+\sigma_n^{p-1}=(\sigma_n-1)^2v+pu$, we have $a^{p^{n+1}}=1$ because $|A_n|=p^n$. Since A_n is cyclic, it follows that $a\in B_{n+1}^{(n)}$ which is a contradiction.

Proof of Proposition. Assume that $D_{r+1}=1$. Then $\mathfrak{p}_{r+1}'^h=(\alpha_{r+1})$ for some $\alpha_{r+1}\in k_{r+1}$. Put $\alpha_r=N_{r+1,r}(\alpha_{r+1})$. Then $\mathfrak{p}_r'^h=(\alpha_r)$ and $\mathfrak{p}^{r+2}$ divides $(N_{r,0}(\alpha_r)^{p-1}-1)$. Hence $n_1^{(r)}=r+2$. Conversely assume that $n_1^{(r)}=r+2$. Let α_r be an element of k_r such that $\mathfrak{p}_r'^h=(\alpha_r)$. It follows that there exists $\alpha_{r+1}\in k_{r+1}$ such that $\alpha_r^{p-1}=N_{r+1,r}(\alpha_{r+1})$ from Lemma 2 and Hasse's norm theorem. Since $N_{r+1,r}(\mathfrak{p}_{r+1}'^{(p-1)h}(\alpha_{r+1}^{-1}))=\mathfrak{p}_r'^{(p-1)h}(\alpha_r^{-1})^{(p-1)}=(1)$, $\mathfrak{p}_{r+1}'^{(p-1)h}(\alpha_{r+1}^{-1})=\mathfrak{a}_{r+1}^{\sigma_r-1}$ for some ideal $\mathfrak{a}_{r+1}$ of k_{r+1}. Thus $D_{r+1}\subset A_{r+1}^{\sigma_r-1}$. Now the assumption (3) and the reflection theorem imply that A_n is cyclic for all $n\geqq 1$. Hence $D_{r+1}=1$ from Lemma 3.

When $p=3$, we calculated $N_{1,0}(E_1)$ and gave some examples of k such that $D_1\neq 1$ in [2]. For those k's with $D_1=1$, we calculated $n_1^{(1)}$ and obtained the following theorem.

Theorem. *Let $p=3$ and $k=\boldsymbol{Q}(\sqrt{m})$ where $m=$*106, 253, 454, 505, 607, 787, 886, 994, 1102, 1294, 1318, 1333, 1462, 1669, 1753, *or* 1810. *Then these k's satisfy all assumptions of proposition and moreover $n_1^{(1)}=2$. Hence $\mu_3(k)=\lambda_3(k)=0$ for the above values of m's.*

Remark. For $m=295$, 397, 745, or 1738, we have $n_1^{(1)}=3$ and $D_2=1$. But the calculation of $n_1^{(2)}$ is difficult since $k_2/\boldsymbol{Q}$ is an extension of degree 18.

References

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