70. *Fourier Coefficients of Certain Eisenstein Series*

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We fix natural numbers $q\geq 3$, k, $n\geq 1$ once and for all. For $\gamma, \delta \in M_n(\boldsymbol{Z})$, we write $(\gamma, \delta)=1$ if $(\gamma\delta)$ is a lower $n\times 2n$ submatrix of some element of $Sp(n;\boldsymbol{Z})$, and put $H_n:=\{z\in M_n(\boldsymbol{C})\mid {}^t z=z, \operatorname{Im} z>0\}$. We fix such a pair γ, δ hereafter. We consider Eisenstein series

$$E(z, s, k;(\gamma, \delta)):=\sum \det(cz+d)^{-k}\operatorname{abs}(\det(cz+d))^{-2s}\quad (z\in H_n,\ s\in \boldsymbol{C}),$$

where (c, d) runs over $G_n(q)\backslash\{(c, d)\mid (c, d)=1,\ c\equiv\gamma,\ d\equiv\delta \bmod q\}$ and $G_n(q)=\{a\in GL_n(\boldsymbol{Z})\mid a\equiv 1_n \bmod q\}$. Our aim is to study Dirichlet series which appear in Fourier coefficients of $E(z, s, k;(\gamma, \delta))$. We denote by $E'(z, s, k;(\gamma, \delta))$ a partial sum of $E(z, s, k;(\gamma, \delta))$ with $\det c\neq 0$. For a ring R, we denote by $\Lambda_n(R)$ the set of all symmetric matrices of degree n with entries in R and put $\Lambda_n:=\Lambda_n(\boldsymbol{Z})$. By Λ'_n we denote the set of all half-integral matrices of degree n, i.e. matrices a such that $2a\in\Lambda_n$ and diagonals of a are integers. Following [3], we put, for $z\in H_n$

$$\sum_{a\in\Lambda_n}\det(z+a)^{-\alpha}\det(\bar{z}+a)^{-\beta}=\sum_{h\in\Lambda'_n}e(\operatorname{tr} hx)\xi(y, h;\alpha, \beta),$$

where $x=\operatorname{Re} z$, $y=\operatorname{Im} z$, $e(w)$ means $\exp(2\pi i w)$ and the function ξ is defined by the above and is fully studied in [3]. We have

$$E'(z, s, k;(\gamma, \delta))=q^{-n(k+2s)}\sum_{h\in\Lambda'_n}\xi(q^{-1}y, h;s+k, s)\zeta(h;k,(\gamma, \delta);s)e(\operatorname{tr} hx/q)$$

where $x=\operatorname{Re} z$, $y=\operatorname{Im} z$ and

$$\zeta(h;k,(\gamma, \delta);s)=\sum_c\sum_d \det(c)^{-k}\operatorname{abs}(\det(c))^{-2s}e(q^{-1}\operatorname{tr} hc^{-1}d).$$

where c runs over $G_n(q)\backslash\{c\in M_n(\boldsymbol{Z})\mid c\equiv\gamma \bmod q,\ \det c\neq 0\}$ and d runs over $\{d\in M_n(\boldsymbol{Z}) \bmod qc\Lambda_n\mid (c, d)=1,\ d\equiv\delta \bmod q\}$. Decompose q as $q=\prod q_i$ where q_i is a power of a prime p_i and for a Dirichlet character χ defined modulo q, we denote by χ_i a Dirichlet character defined modulo q_i such that $\chi=\prod\chi_i$. Then we have

$$\zeta(h;k,(\gamma, \delta);s)=2\varphi(q)^{-1}\sum_{\substack{\chi \bmod q\\ \chi(-1)=(-1)^k}}\prod_{p\nmid q} b_p((p^{k+2s}\chi(p))^{-1}, h)$$
$$\times\prod_i b_{p_i}((p_i^{k+2s}(\prod_{j\neq i}\chi_j)(p_i))^{-1};h,\chi,(\gamma, \delta)),$$

where φ is the Euler's function and we put, for $h\in\Lambda'_n$

$$b_p(x, h)=\sum_{r\in\Lambda_n(\boldsymbol{Q}_p)/\Lambda_n(\boldsymbol{Z}_p)}x^{\operatorname{ord}_p\nu(r)}e(\operatorname{tr} hr),$$

where $\nu(r)$ is the product of reduced denominators of elementary divisors of r. To define the function b_{p_i}, we put, for a power Q of a prime p, $h\in\Lambda'_n$ and a Dirichlet character χ defined modulo Q,

$$B_p(x;h,\chi;(\gamma, \delta), Q)=\sum_{c\in U_n\backslash c(n;p)}x^{\operatorname{ord}_p\det c}\sum_{\substack{d \bmod Qc\Lambda_n\\ c^t d\in\Lambda_n}}e(Q^{-1}\operatorname{tr} hc^{-1}d)\sum_g\chi(\det g),$$

where g runs over $GL_n(\boldsymbol{Z}/Q\boldsymbol{Z})$ with $c\equiv g\gamma \bmod Q$ and $d\equiv g\delta \bmod Q$ (as a matter of fact, the possibility of g is at most one), and $U_n=SL_n(\boldsymbol{Z}_p)$, $c(n;p)=\{u\in M_n(\boldsymbol{Z}_p)\mid \det u \text{ is a power of } p\}$. Then $b_{p_i}(x;h,\chi,(\gamma,\delta))=B_{p_i}(x;h,\chi_i^{-1};(\gamma,q_i'\delta),q_i)$ where q_i' is an integer such that $(qq_i^{-1})q_i'\equiv 1 \bmod q_i$.

On $b_p(x,h)$, we know ([1]) the following

Theorem. (i) *$b_p(x,0_n)=(1-x)\prod_{0<k\le[n/2]}(1-p^{2k}x^2)\{(1-p^n x)\prod_{\substack{n+1\le j<2n\\ 2\nmid j}}(1-p^j x^2)\}^{-1}$, where $[a]$ denotes the largest integer which does not exceed a.*

(ii) *Let $h=\begin{pmatrix} h_1 & 0\\ 0 & 0\end{pmatrix}\in \Lambda_n(\boldsymbol{Z}_p)$ for $h_1\in\Lambda_r(\boldsymbol{Z}_p)$ with $\det h_1\neq 0$ $(1\le r\le n)$.*

If r is odd, then

$$b_p(x,h)=f(x)(1-x)\prod_{1\le j\le[n/2]}(1-p^{2j}x^2)\{\prod_{\substack{n+1\le k\le 2n-r\\ 2\nmid k}}(1-p^k x^2)\}^{-1}.$$

If r is even, then

$$b_p(x,h)=g(x)(1-x)\prod_{1\le j\le[n/2]}(1-p^{2j}x^2)\{(1-\eta p^{n-r/2}x)\prod_{\substack{n+1\le k\le 2n-r\\ 2\nmid k}}(1-p^k x^2)\}^{-1}.$$

Here $f(x)$, $g(x)$ are polynomials in x and η is 0 or ± 1.

If p does not divide $\det 2h_1$, then $f(x)=g(x)=1$ and $\eta=\left(\frac{(-1)^{r/2}\det 2h_1}{p}\right)$ (Kronecker symbol).

Theorem. *Let p be a prime and Q a power of p and for a Dirichlet character χ defined modulo Q, $h\in\Lambda_n'$ and $(\gamma,\delta)=1$, following assertions are true.*

(i) *$B_p(x;h,\chi;(\gamma,\delta),Q)$ is a rational function in x whose (not necessarily reduced) denominator is*

$$(1-p^t x)\prod_{\substack{t+1\le j\le 2n-r\\ 2\nmid j}}(1-p^j x^2)\prod_{\substack{2t+1\le i\le 2n-r\\ 2\mid i}}(1-p^i x^2)$$

where $r=\operatorname{rank} h$, $t=n-r$.

(ii) *If $\det h\neq 0$ or $\chi^2\neq id$, then $B_p(x;h,\chi;(\gamma,\delta),Q)$ is a polynomial in x.*

(iii) *If $\chi^2\neq id$ and there is an elementary divisor of γ which is divided by Q, then $B_p(x;0_n,\chi;(\gamma,\delta),Q)=0$.*

(iv) *Let $\gamma\equiv u\begin{pmatrix}0 & 0\\ 0 & \gamma_4\end{pmatrix}{}^t v$, $\delta\equiv u\begin{pmatrix}\delta_1 & 0\\ 0 & \delta_4\end{pmatrix}v^{-1} \bmod Q$, where $u,v\in U_n$, $\gamma_4,\delta_4\in M_t(\boldsymbol{Z}_p)$, $\delta_1\in M_r(\boldsymbol{Z}_p)$ $(r+t=n)$ and assume $\det\gamma_4\neq 0$ and $Q\gamma_4^{-1}\equiv 0 \bmod p$. Then we have*

$$B_p(x;0_n,\chi;(\gamma,\delta),Q)=\bar\chi(\det\delta_1)\bar\chi(\det(\gamma_4\tilde\gamma_4^{-1}))(\det\tilde\gamma_4)^r \times B_p(x;0_t,\chi;(\tilde\gamma_4,\tilde\delta_4),Q)B_p(p^t x;0_r,\chi;(0_r,1_r),Q),$$

where $\tilde\gamma_4$ is the elementary divisor matrix of γ_4 and $\tilde\delta_4$ is defined by $\gamma_4\equiv u_0\tilde\gamma_4 v_0 \bmod Q$, $\delta_4\equiv u_0\tilde\delta_4{}^t v_0^{-1} \bmod Q$ for $u_0\in GL(t,\boldsymbol{Z}/q\boldsymbol{Z})$, $v_0\in U_t$. If $r=0$, then we must put $\bar\chi(\det\delta_1)B_p(p^t x;0_r,\chi;(0_r,1_r),Q)=1$. If $t=0$, then we put $\bar\chi(\det(\gamma_4\tilde\gamma_4^{-1}))(\det\tilde\gamma_4)^r B_p(x;0_t,\chi;(\tilde\gamma_4,\tilde\delta_4),Q)=1$.

(v) *If $\det\gamma\neq 0$ and $Q\gamma^{-1}\equiv 0_n \bmod p$, then we have*

$$B_p(x;0_n,\chi;(\gamma,\delta),Q)=\bar\chi(p^{-\operatorname{ord}_p\det\gamma}\det\gamma)x^{\operatorname{ord}_p\det\gamma}\sum_{A\in\Lambda_n/\gamma\Lambda_n{}^t\gamma}\chi(\det(1+A\alpha Q\gamma^{-1}))$$

where α is any element such that $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in Sp(n,\boldsymbol{Z}/Q\boldsymbol{Z})$ for some β.

(vi) *Suppose $\chi^2=id$, $\chi\neq id$; then $B_p(x;0_n,\chi;(0_n,1_n),Q)=0$ either for $p\neq 2$ and odd n or for $p=2$.*

(vii) $B_p(x; 0_n, \chi; (0_n, 1_n), Q) = Q_2^{n(n+1)} x^{n \operatorname{ord}_p Q_2} B_p(x; 0_n, \chi; (0_n, 1_n), Q_1)$ *where* $Q_1 = \operatorname{lcm}(p,$ *conductor of* $\chi)$ *and* $Q_2 = Q/Q_1$, *and we have*

$$B_p(x; 0_n, id; (0_n, 1_n), p) = p^{n(n+1)} \prod_{\substack{1 \le j \le n \\ 2 \nmid j}} (1 - p^{-j}) x^n \times \{(1 - p^n x) \prod_{\substack{n+1 \le j < 2n \\ 2 \nmid j}} (1 - p^j x^2)\}^{-1} \begin{cases} 1 & 2 \nmid n, \\ 1 - x & 2 \mid n. \end{cases}$$

If $\chi^2 = id$, $\chi \neq id$, $2 \mid n$ *and* $p \neq 2$, *then*

$$B_p(x; 0_n, \chi; (0_n, 1_n), p) = p^{n(n+1)} (\chi(-1) p)^{n/2} \prod_{\substack{1 \le j \le n \\ 2 \nmid j}} (1 - p^{-j}) x^n \{ \prod_{\substack{n+1 \le j < 2n \\ 2 \nmid j}} (1 - p^j x^2) \}^{-1}.$$

The denominator of $B_p(x; h, \chi; (\gamma, \delta), Q)$ in (i) seems to be too big, as contrasted with the previous theorem.

To prove the theorem, the following are necessary.

Lemma. *Let* $r + t = n$, $r \geq 1$, $t \geq 1$. *Then a complete set of representatives of* $U_n \backslash \{(c, d) \mid c\,{}^t d \in \Lambda_n,\ c \in c(n; p),\ d \in M_n(\mathbf{Z}_p)\}$ *is*

$$c = \begin{pmatrix} w\gamma & 0 \\ -c_4{}^t d_2{}^t w^{-1} \alpha + f\gamma & c_4 \end{pmatrix}, \quad d = \begin{pmatrix} w\delta & d_2 \\ -c_4{}^t d_2{}^t w^{-1t} \beta + f\delta & d_4 \end{pmatrix}$$

where w, γ *run over* $U_r \backslash c(r; p)$, c_4 *does over* $U_t \backslash c(t; p)$, δ *does over* $M_r(\mathbf{Z}_p)$ *with* $(\gamma, \delta) = 1$, f *does over* $M_{t,r}(\mathbf{Z}_p)/M_{t,r}(\mathbf{Z}_p) w$, d_2 *does over* $M_{r,t}(\mathbf{Z}_p)$ *with* $c_4{}^t d_2{}^t w^{-1}$ *integral and* d_4 *does over* $M_t(\mathbf{Z}_p)$ *with* $c_4{}^t(d_4 - f w^{-1} d_2) \in \Lambda_t(\mathbf{Z}_p)$. *Moreover* α, β *are arbitrarily fixed matrices so that* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sp(r; \mathbf{Z}_p)$.

Lemma. *Let* $h \in \Lambda_n'$ *with* $\det h \neq 0$, $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, *and* Q *a power of a prime* p. *If* λ_n *is sufficiently large (dependently on* Q *and* h*), then*

$$\sum_c \sum_d e(Q^{-1} \operatorname{tr} h c^{-1} d) = 0$$

where c *runs over* $U_n(Q) \backslash \{$*the set of* $c \in c(n; p)$ *such that* $\{p^{\lambda_1}, \cdots, p^{\lambda_n}\}$ *is elementary divisors of* $c\}$, *and* d *runs over* $\{d \in M_n(\mathbf{Z}_p) \bmod Qc\Lambda_n(\mathbf{Z}_p) \mid c^t d \in \Lambda_n(\mathbf{Z}_p),\ (c, d) \equiv (\gamma, \delta) \bmod Q\}$. *Here we put* $U_n(Q) = \{u \in SL_n(\mathbf{Z}_p) \mid u \equiv 1_n \bmod Q\}$.

To evaluate $B_p(x; 0_n, \chi; (0_n, 1_n), p)$, we need a well known formula for Gaussian polynomials. Put

$$(q)_n = \begin{cases} 1 & \text{if } n = 0, \\ \prod_{i=1}^n (1 - q^i) & \text{if } n > 0, \end{cases}$$

and

$$\begin{bmatrix} m \\ n \end{bmatrix} = \begin{cases} (q)_m (q)_n^{-1} (q)_{m-n}^{-1} & \text{if } 0 \leq n \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Then the following is useful.

Lemma.

$$\prod_{i=0}^{n-1} (X - q^i Z) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \prod_{j=0}^{k-1} (Y - q^j Z) \prod_{h=0}^{n-k-1} (X - q^h Y).$$

Details will appear elsewhere.

References

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