68. A Remark on $B(P, \alpha)$-refinability

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Introduction. Recently a number of general topological properties have been introduced in order to obtain covering characterizations of generalized normal and paracompact spaces. In particular see [1, 2, 7, 10] for such characterizations of subparacompact, θ-refinable, collectionwise normal and collectionwise subnormal spaces. In this paper we consider the general property of $B(P, \alpha)$-refinable and show how this notion is used to generalize known results for normal and collectionwise normal spaces.

The union of any family $\mathcal{U}$ will be denoted by $\mathcal{U}^*$. The properties P considered in this paper will be discrete (D), locally finite (LF) and closed (C). Countable ordinals will be denoted by λ and α will be any ordinal.

Definition 1. A space X is *$B(P, \alpha)$-refinable* provided every open cover $\mathcal{U}$ of X has a refinement $\mathcal{E}=\cup\{\mathcal{E}_\beta : \beta<\alpha\}$ which satisfies i) $\{\cup\mathcal{E}_\beta : \beta<\alpha\}$ partitions X, ii) for every $\beta<\alpha$, $\mathcal{E}_\beta$ is a relatively P collection of closed subsets of the subspace $X-\cup\{\cup\mathcal{E}_\mu : \mu<\beta\}$, and iii) for every $\beta<\alpha$, $\cup\{\cup\mathcal{E}_\mu : \mu<\beta\}$ is a closed set. For the case $P=C$, we require $\mathcal{E}_\beta$ to be a one-to-one partial refinement of $\mathcal{U}$ for each $\beta<\alpha$.

The collection $\mathcal{E}$ is often called a *$B(P, \alpha)$-refinement* of $\mathcal{U}$.

In [6, 7] the author has used the property of weakly $\bar{\theta}$-refinable to obtain several open cover characterizations for normal and collectionwise normal spaces. The following are modifications of this idea.

Definition 2. An open cover $\mathcal{G}=\cup\{\mathcal{G}_n : n\in N\}$ of a space X is a (k^-) *bded-weak $\bar{\theta}$-cover* if (i) the collection $\{\mathcal{G}_n^* : n\in N\}$ is point finite and (ii) for each n, there exist an integer $k(n)$ ($\leq k$) such that $X=\{x : 0<ord(x, \mathcal{G}_n)\leq k(n), n\in N\}$. Spaces for which each open cover has a refinement with the above property are called *(k^-)-bded-weak $\bar{\theta}$-refinable.*

Remark. A k-bded weak $\bar{\theta}$-cover is equivalent to a boundly weak $\bar{\theta}$-cover, as defined in [10].

Main results.

Theorem 1. *A space X is bded-weak $\bar{\theta}$-refinable iff X is 1-bded weak $\bar{\theta}$-refinable.*

Proof. The sufficiency is clear. Let $\mathcal{G}=\{\mathcal{G}_n : n\in N\}$ be a bded-weak $\bar{\theta}$-cover of X with $k(n)$ defined as above.

For each $x\in X$ and every $n, j\in N$, define $W(n, x)=\cap\{G\in\mathcal{G}_n : x\in G\}$, and $\mathcal{W}(n, j)=\{W(n, x) : ord(x, \mathcal{G}_n)=j\}$ so that if $ord(x, \mathcal{G}_n)=j$, then $ord(x, \mathcal{W}(n, j))=1$. Define $\mathcal{W}=\cup\{\mathcal{W}(n, j) : 0<j\leq k(n), n\in N\}$. It should

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be clear that $\mathcal{W}$ is an open refinement of $\mathcal{G}$, and $X=\{x: ord(x, \mathcal{W}(n, j))=1, 0<j\leq k(n), n\in N\}$. Furthermore, for each $x\in X$, there exists an integer M such that $x\notin\{\cup\mathcal{G}_n: n>M\}$ so that $x\notin\{\cup\mathcal{W}(n, j): n>M\}$. Therefore, $\{\cup\mathcal{W}(n, j): 0<j\leq k(n), n\in N\}$ is point finite and the proof is complete.

Theorem 2. *A space X is $B(D, \omega)$-refinable iff X is bded-weak $\bar{\theta}$-refinable.*

Proof. (i) Let $\mathcal{U}$ be an open cover of X with $B(D, \omega)$-refinement $\mathcal{E}=\cup\{\mathcal{E}_n=\{E(\alpha, n): \alpha\in A\}: n\in N\}$. For each $\alpha\in A$ and $n\in N$, choose $U(\alpha, n)\in\mathcal{U}$ such that $E(\alpha, n)\subset U(\alpha, n)$, and define

$$G(\alpha, n)=U(\alpha, n)-\cup\{E(\beta, n): \beta\neq\alpha\}-\cup\{\cup\mathcal{E}_k: k<n\},$$
$$\mathcal{G}_n=\{G(\alpha, n): \alpha\in A\}, \quad \text{and}$$
$$\mathcal{G}=\cup\{\mathcal{G}_n: n\in N\}.$$

It is easy to see that $\mathcal{G}$ is a 1-bded-weak $\bar{\theta}$-refinement of $\mathcal{U}$.

(ii) Let $\mathcal{G}=\cup\{\mathcal{G}_n: n\in N\}$ be a 1-bded-weak $\bar{\theta}$-cover of X. We construct a $B(D, \omega)$-refinement of $\mathcal{G}$. Now

(1) Let $\mathcal{G}^*=\{\cup\mathcal{G}_n: n\in N\}$, a point finite collection.

(2) For each $n\in N$, define $C_n=\{x: ord(x, \mathcal{G}^*)=n\}$.

(3) For each $n\in N$, define $F_n=\{f: \{1, 2, \cdots, n\}\to N, f(1)<f(2)<\cdots<f(n)\}$.

(4) For each $n\in N$ and $x\in C_n$, let f_x represent the unique member of F_n such that $x\in W(x)$, where $W(x)=\cap\{\cup\mathcal{G}_{f_x(r)}: 1\leq i\leq n\}$.

I. By induction, for each $n\in N$ we construct a family $\mathcal{H}_n=\cup\{\mathcal{H}(n, m): 1\leq m\leq n\}$ of collections of sets such that

(a_1) $\mathcal{H}(n, m)$ is a partial refinement of $\mathcal{G}$ for $1\leq m\leq n$,

(a_2) $C_n=\cup\{\cup\mathcal{H}(n, m): 1\leq m\leq n\}$ for each $n\in N$,

(a_3) for $1\leq m\leq n$, $(\cup\mathcal{H}(n, m))\cap E(n, m)=\emptyset$, where $E(n, m)=\cup\{C_k: k<n\}\cup(\cup\mathcal{H}(n, r): 1\leq r<m\})$, and

(a_4) $\mathcal{H}(n, m)$ is a relatively discrete collection of closed subsets of the subspace $X-E(n, m)$ for $1\leq m\leq n$.

For $n=1$, define $\mathcal{H}(1, 1)=\{C_1\cap G: G\in\mathcal{G}\}$. Now $E(1, 1)=\emptyset$. It should be clear that $\mathcal{H}(1, 1)$ satisfies conditions (a_1)–(a_3) above. We assert that $\mathcal{H}(1, 1)$ is a discrete collecton of closed subsets of X and hence satisfies (a_4). Indeed, let $x\in X$. If $x\in C_k$ for some $k>1$, then there exist two members of $\mathcal{G}^*$ which contain x and whose intersection is a neighborhood of x that misses C_1 and hence misses $\cup\mathcal{H}(1, 1)$. If $x\in C_1$, then $x\in C_1\cap G$ for some $G\in\mathcal{G}$. It is easy to check that G is a neighborhood of x that misses every member of $\mathcal{H}(1, 1)$ except $C_1\cap G$.

Now let n be fixed and assume that $\mathcal{H}_k$ has been constructed such that $\mathcal{H}_k$ satisfies (a_1)–(a_4) above for each k, $1\leq k<n$. We construct $\mathcal{H}_n$. For each $k\in N$ and $1\leq m\leq n$, define $C(n, m, k)=\{x\in C_n: m=min(\{r: ord(x, \mathcal{G}_{f_x(r)})=1\})$, and $f_x(m)=k\}$,

$\mathcal{H}(n, m, k)=\{C(n, m, k)\cap G: G\in\mathcal{G}_k\}$, $\mathcal{H}(n, m)=\cup\{\mathcal{H}(n, m, k): k\in N\}$, and $\mathcal{H}=\cup\{\mathcal{H}(n, m): 1\leq m\leq n\}$.

The following properties are easy to verify.

(i) $C(n, m, k)=\cup\mathcal{H}(n, m, k)$ for each $k\in N$ and $1\leq m\leq n$.

(ii) If $(n, m, k) \neq (r, s, t)$, then $C(n, m, k) \cap C(r, s, t) = \emptyset$. In particular, $[\cup \mathcal{H}(n, m, k)] \cap [\cup \mathcal{H}(r, s, t)] = \emptyset$.

(iii) If $j \neq k$ and $x \in C(n, m, k)$, then $W(x)$ is a neighborhood of x such that $W(x) \cap C(n, m, j) = \emptyset$. In particular, $W(x) \cap (\cup \mathcal{H}(n, m, j)) = \emptyset$ Indeed if $y \in C(n, m, j)$, then $f_y(m) = j \neq k = f_x(m)$; hence, $\{\mathcal{G}_{f_y(i)} : 1 \leq i \leq n\} \neq \{\mathcal{G}_{f_x(i)} : 1 \leq i \leq n\}$. Since $ord(y, \mathcal{G}^*)j = n$, it thus follows that $y \notin W(x)$.

The fact that $\mathcal{H}_n$ (a_1)–(a_4) above is straightforward and left for the reader.

II. Define a well-order "$<$" on the set $S = \{(n, m) : 1 \leq m \leq n, n \in N\}$ such that for every $(n, m), (k, r) \in S$,

$$(n, m) < (k, r) \quad \text{iff} \quad \left\{ \begin{matrix} n < k \text{ or} \\ n = k \text{ and } m < n \end{matrix} \right\}.$$

Let $g : S \to N$ be the unique bijection which preserves this order.

For each $n \in N$, define $\mathcal{F}_n = \mathcal{H}(k, r)$ such that $g(k, r) = n$, and $\mathcal{F} = \cup \{\mathcal{F}_n : n \in N\}$.

From the fact that $X = \cup \{C_n : n \in N\}$ and that $\mathcal{H}(n,m)$ satisfies conditions (a_1)–(a_4) above for every $n \in N$ and $1 \leq m \leq n$, it is easy to see that $\mathcal{F}$ is a $B(D, \omega)$-refinement of $\mathcal{G}$.

Remark. (1) It has been shown [8] that every θ-refinable space is $B(D, \omega)$-refinable.

(2) It is stated in [10] that Long Bing [4] has independently obtained the sufficiency of Theorem 2 above.

In [6], the author showed that normality is equivalent to every weak $\bar{\theta}$-cover having a closed shrink. We now have a generalization of this result.

Theorem 3. *A space X is normal iff every open cover of X which has a $B(C, \lambda)$-refinement also has a closed shrink.*

Proof. The sufficiency is clear so let X be normal and $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ an open cover of X which has a $B(C, \lambda)$-refinement $\mathcal{E} = \cup \{\mathcal{E}_\gamma = \{E(\gamma, \alpha) : \alpha \in A\} : \gamma < \lambda\}$. By transfinite induction we construct for every $\gamma < \lambda$, a collection $\mathcal{H}_\gamma = \{H(\gamma, \alpha) : \alpha \in A\}$ of cozero subsets of X satisfying

(i) $H_\gamma^* = \cup \mathcal{H}_\gamma$ is a cozero set, and

(ii) $F(\gamma, \alpha) = (E(\gamma, \alpha) - \cup \{H_\beta^* : \beta < \gamma\}) \subset H(\gamma, \alpha) \subset cl(H(\gamma, \alpha)) \subset U_\alpha$ for every $\alpha \in A$.

For fixed $\gamma < \lambda$ assume that the collections $\mathcal{H}_\beta$ with the above properties have been constructed for all $\beta < \gamma$. Now $\cup \{H_\beta^* : \beta < \gamma\}$ is an open set which by condition (ii) above contains $\cup \{\cup \mathcal{E}_\beta : \beta < \gamma\}$; hence, $\{F(\gamma, \alpha) : \alpha \in A\}$ is a collection of closed subsets of X such that $F^* = \cup \{F(\gamma, \alpha) : \alpha \in A\}$ is a closed set. Also, $F(\gamma, \alpha) \subset U_\alpha$ for each $\alpha \in A$. Since X is normal there exists a cozero set $H(\gamma, \alpha)$ such that $F(\gamma, \alpha) \subset H(\gamma, \alpha) \subset cl(H(\gamma, \alpha) \subset U_\alpha)$ where H^* is a cozero set, and the construction is complete. Now by Theorem 4.3 of [8] it follows that $\mathcal{U}$ has a closed shrink.

Corollary. *Let X be a normal space.*

(i) *If X is B(C, λ)-reflnable, then every open cover of X has a closed shrink.*

(ii) *If X is countably B(C, λ)-reflnable, then every countable open cover of X has a closed shrink.*

(iii) *X is countably paracompact iff X is countably B(C, λ)-refinable.*

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