67. On a Generalization of MacPherson's Chern Homology Class

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§0. **Introduction.** Unlike in the non-singular case, there is no general notion available of characteristic classes of singular complex algebraic varieties, except for Deligne-Grothendieck-MacPherson's theory C_* (abbr. *DGM-theory*) of Chern class [3] and Baum-Fulton-MacPherson's theory Td_* (abbr. *BFM-theory*) of Todd class [1]. C_* and Td_* are both formulated as unique *natural transformations* from *certain group functors* to the *homology group functor* such that they satisfy certain "*smooth condition*" (see below). In this note we show that DGM-theory can be generalized to other "Chern-type" characteristic classes. This work is motivated by R. MacPherson's survey article [4] and more details of this work will be treated in [5].

§1. **DGM-theory** ([3, 4]). Let $\mathcal{CV}$ be the category of compact complex algebraic varieties and $\mathcal{Ab}$ be the category of abelian groups. Let $\mathcal{F}: \mathcal{CV} \to \mathcal{Ab}$ be the "constructible function" (covariant) functor such that for $X \in \mathrm{Obj}(\mathcal{CV})$ $\mathcal{F}(X)$ is the abelian group of constructible functions on X. Let $H_*(\ \ ,\boldsymbol{Z})$ be the usual $\boldsymbol{Z}$-homology group functor. Then
DGM-theory:

(1) there exists a *unique natural transformation*

$$C_*: \ \mathcal{F} \longrightarrow H_*(\ \ ,\boldsymbol{Z}),$$

such that

(2) ("*smooth condition*") if X is smooth, then $C_*(X)(1_X)=c(TX)\cap[X]$, where 1_X is the characteristic function of X and $c(TX)$ is the usual *total* cohomology Chern class of the tangent bundle TX.

In passing, analogously, *BFM-theory* Td_* of Todd class is formulated as follows: Let $K_*: \mathcal{CV} \to \mathcal{Ab}$ be the "coherent sheaf" group functor such that for $X \in \mathrm{Obj}(\mathcal{CV})$ $K_*(X)$ is the Grothendieck group of algebraic coherent sheaves on X, and let $H_*(\ \ ,\boldsymbol{Q})$ be the usual $\boldsymbol{Q}$-homology group functor. Then BFM-theory says that (1) there exists a *unique natural transformation* $Td_*: K_* \to H_*(\ \ ,\boldsymbol{Q})$ such that (2) ("*smooth condition*") if X is smooth, then $Td_*(X)(I_X)=td(TX)\cap[X]$, where I_X is the trivial line bundle over X and $td(TX)$ is the usual *total* cohomology Todd class of the tangent bundle TX.

§2. **A generalization of DGM-theory.** It should be emphasized that DGM-theory C_* of (total) Chern homology class and BFM-theory Td_* of

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(total) Todd homology class are the analogy of the following classical counterparts:

total Chern class of vector bundles: $c=1+\sum_{i\geq 1} c_i : K \longrightarrow H^*(\ \ ,Z)$,

total Todd class of vector bundles: $td=1+\sum_{i\geq 1} td_i : K \longrightarrow H^*(\ \ ,Q)$,

are *natural transformations*, where K is the Grothendieck group functor. For a generalization of DGM-theory, let us consider the Chern polynomial theory $c_t := 1+\sum_{i\geq 1} c_i t^i : K \to H^*(\ \ ,Z)[t] := H^*(\ \ ,Z)\otimes_Z Z[t]$. If we "evaluate" c_t at various integers, then we get various "Chern-type" characteristic classes from K to $H^*(\ \ ,Z)$. In particular, if we "evaluate" c_t at $t=1$, then we get the above total Chern class theory, which has a singular version, i.e., DGM-theory.

Theorem: *Let* $\mathcal{F}^t : \mathcal{V} \to \mathcal{Ab}$ *be the correspondence such that for* $X \in \mathrm{Obj}(\mathcal{V})$ $\mathcal{F}^t(X) := \mathcal{F}(X)\otimes_Z Z[t]$, *where* $\mathcal{F}(X)$ *is the abelian group of constructible functions on* X *as in DGM-theory. Let* $H_*(\ \ ,Z)[t] : \mathcal{V} \to \mathcal{Ab}$ *be the usual* $Z[t]$*-homology group funcor. Then*

(1) *the correspondence* $\mathcal{F}^t$ *can be made a covariant functor such that when* $t=1$ $\mathcal{F}^t$ *is nothing but DGM*'s *constructible function functor* $\mathcal{F}$ *(see Remarks below),*

with this "$Z[t]$*-constructible function" functor* $\mathcal{F}^t$,

(2) *there exists a unique natural transformation*

$$C_{t*} : \mathcal{F}^t \longrightarrow H_*(\ \ ,Z)[t],$$

such that

(3) (*"smooth condition"*) *if* X *is smooth, then* $C_{t*}(X)(1_X) = c_t(TX) \cap [X]$, *where* 1_X *is the characteristic function of* X *and* $c_t(TX)$ *is the cohomology Chern polynomial of the tangent bundle* TX, *and*

(4) ($C_{1*}=$*DGM-theory* C_*) *if we "evaluate"* C_{t*} *at* $t=1$, *then we get DGM-theory* C_*.

Remarks: (i) In the above theorem we *cannot* replace $\mathcal{F}^t$ by DGM's functor $\mathcal{F}$. This can be easily seen by considering a simple situation where $f : X \to pt$ is a map from a smooth variety X to a point. (ii) As a functor $\mathcal{F}^t$ *cannot* be a linear extension of DGM's functor $\mathcal{F}$ with respect to $Z[t]$. These two points make the theorem non-trivial. (iii) Unlike in the classical counterpart, we *cannot* express $C_{t*}=\sum_{i\geq 0} P_i(t) C_{*i}$, where $P_i(t) \in Z[t]$ and $C_{*i} : \mathcal{F} \to H_{2i}(\ \ ,Z)$ is the composite of DGM-theory $C_* : \mathcal{F} \to H_*(\ \ ,Z)$ and $H_*(\ \ ,Z) \to H_{2i}(\ \ ,Z)$, the natural transformation "picking up" the $2i$-dimensional homology classes.

Corollary: *Let* w *be any non-zero integer and consider a "Chern-type" characteristic class of vector bundles* $c_w := 1+\sum_{i\geq 1} w^i c_i : K \to H^*(\ \ ,Z)$. *Then*

(1) *there exists a unique natural transformation*

$$C_{w*} : \mathcal{F}^w \longrightarrow H_*(\ \ ,Z)$$

such that

(2) (*"smooth condition") if X is smooth, then* $C_{w*}(X)(1_X)=c_w(TX)\cap[X]$. (*Here we note that as correspondences* $\mathcal{F}^w$ *and* $\mathcal{F}$ *are the same, i.e.,* $\mathcal{F}^w(X)=\mathcal{F}(X)$, *but as functors they are quite different* (*see* § 3).)

§ 3. A possible connection with $\mathcal{D}$-module theory. Our $Z[t]$-constructible function functor $\mathcal{F}^t$ has the following simple but interesting push-forward property: Let $f: X\to Y$ be a morphism. If, under the DGM's functor $\mathcal{F}$, $\mathcal{F}(f)(Eu_W)=\sum_S n_S Eu_S\in\mathcal{F}(Y)$, where Eu_W is MacPherson's *local Euler obstruction*, then $\mathcal{F}^t(f)(Eu_W)=\sum_S n_S t^{\dim W-\dim S}Eu_S$. Thus, if we consider our C_{-1*}, i.e., a "singular version" of the total Chern class $c_{-1}=1+\sum_{i\geq 1}(-1)^i c_i$ of the *dual* of vector bundles, then

$$\mathcal{F}^{-1}(f)(Eu_W)=\sum n_S(-1)^{\dim W-\dim S}Eu_S.$$

This kind of constructible function involving "twisting" appears in Kashiwara's local index theorem for a holonomic $\mathcal{D}$-module $\mathcal{M}$ [2]:

$$\chi_{\mathcal{M}}=\sum m_\alpha(-1)^{\operatorname{codim} Z_\alpha}Eu_{Z_\alpha},$$

where $\mathcal{M}$ is a holonomic $\mathcal{D}$-module on X and $Ch(\mathcal{M})\subset\bigcup T^*_{Z_\alpha}X$ and m_α is the multiplicity of $T^*_{Z_\alpha}X$ in $Ch(\mathcal{M})$. So, in connection with our C_{-1*}, a naive question is whether or not one could find a smooth manifold $M\supset X$ and a morphism $f: M\to X$ such that

$$\mathcal{F}^{-1}(f)(1_X)=\mathcal{F}^{-1}(f)(Eu_X)=\sum_\alpha m_\alpha(-1)^{\dim X-\dim Z_\alpha}Eu_{Z_\alpha}=\chi_{\mathcal{M}}.$$

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