64. *Topological Aspects of Conformally Flat Manifolds*

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1. Introduction. This is a research announcement concerning foundations of conformally flat manifolds. We assume throughout that M be a *closed* oriented smooth n-dimensional manifold and any map or transformation which appears in the sequal is orientation preserving.

A Riemannian manifold (M, g) is called *conformally flat* if for any $x \in M$, there exist a neighbourhood U of x and a smooth embedding $\phi: U \to S^n$ such that $\phi^* g_S = \mu \cdot g$, where g_S is the spherical metric of S^n and μ is a positive valued continuous function on U.

Recall Liouville's theorem: any (locally defined) conformal map of S^n is the restriction of a Moebius transformation, provided $n \geq 3$. Thus the above ϕ is unique up to the composition with a Moebius transformation, if $n \geq 3$. This quickly yields a system of local charts of M modelled on S^n with transition functions Moebius transformations. Further, by means of analytic continuation, we get a developing map $D: \tilde{M} \to S^n$ and a holonomy homomorphism $h: \pi_1(M) \to Mob^+(S^n)$, where $\tilde{M}$ is the universal covering of M and $Mob^+(S^n)$ is the group of all the orientation preserving Moebius transformations of S^n. They satisfy $D(\gamma x) = h(\gamma) D(x)$, where $\gamma \in \pi_1(M)$ and $x \in \tilde{M}$. The image of h is called the holonomy group and denoted by Γ.

In dimension 2, by a conformally flat structure we mean the structure given by the pair of a developing map and a holonomy homomorphism, i.e. the geometric structure known as *projective structure*.

Examples of conformally flat structures are usually constructed as follows. Let $\Gamma \subset Mob^+(S^n)$ be a discrete subgroup which acts freely and properly discontinuously on a Γ-invariant domain U of S^n. Then $M = U/\Gamma$ carries naturally a conformally flat structure. However examples are known of conformally flat manifolds whose developing maps are not covering maps. ([3], [6], [7], [9])

In § 2, given a conformally flat manifold M, we define its limit set, a subset of S^n, and give criterions for the developing map to be a covering map. In § 3, we describe conditions for M to have a finite limit set. § 4 is devoted to the study of the case where the limit set is a Cantor set.

Details including full proofs will appear elsewhere.

2. Limit set. We define the limit set of M in four different ways and show that they all coincide. Recall that Moebius transformations on S^n are extended in a canonical way to transformations on D^{n+1} and that they preserve the Poincaré metric on $IntD^{n+1}$. If they are not the identity, Moebius transformations are classified into three cases.

1. *Elliptic* transformations, which have fixed points in $IntD^{n+1}$.

2. *Parabolic* transformations, which have no fixed points in $IntD^{n+1}$ and exactly one fixed point in S^n.

3. *Loxodromic* transformations having no fixed points in $IntD^{n+1}$ and exactly two fixed points in S^n.

Now we shall give definitions of the limit set.

Definition. (1) L_F is the closure of the set of fixed points of loxodromic or parabolic elements of the holonomy group Γ.

(2) L_ω is the set of accumulation points in S^n of the Γ-orbit of some point a in $IntD^{n+1}$.

(3) L_J is the set of points x such that for any neighbourhood U of x in S^n, the family $\{\gamma|_U\}_{\gamma\in\Gamma}$ is not equicontinuous.

(4) L_O is the set of points x such that for any compact neighbourhood U of x, $D^{-1}(U)$ has a noncompact component.

The difference of the Euclidean metric on D^{n+1} and the Poincaré metric on $IntD^{n+1}$ shows that L_ω is independent of the particular choice of a. Since D is a submersion, L_O is equal to the set of points not evenly covered by D.

Theorem 1. $L_F = L_\omega = L_J = L_O$.

That $L_O \subset L_J$ is already known by Kulkarni and Pinkall ([8]). Hereafter we shall simply denote the set in Theorem 1 by L and call it the limit set. Clearly L is closed and Γ-invariant. By a standard argument for L_ω, we get the following proposition.

Proposition 2. *Let Λ be a Γ-invariant closed subset of S^n which is not a singleton. Then $L \subset \Lambda$.*

As an application, we have:

Corollary 3. *If D is not surjective, then D is a covering map onto its image.*

Corollary 3 is originally due to Kamishima ([5]). Our proof here is a direct application of Theorem 1 and Proposition 2 and is elementary. Notice however the case where $S^n - Image(D)$ is a singleton reduces to Fried's theorem ([2]).

Another criterion in terms of the topology of the limit set is in order.

Corollary 4. *Suppose* (1) *and* (2) *below.*

(1) *$S^n - L$ is connected and simply connected.*

(2) *Any point of L has an arbitrarily small neighbourhood V such that $V - L$ is connected.*

Then the developing map is a covering map onto its image.

There exist examples in dimension two showing that the condition (2) is in fact neccessary.

3. Elementary conformaly flat manifolds. It is easy to show that if the cardinality of the limit set L is greater than 2, then L is a perfect set.

We call M *elementary* in case L is a finite set. They are classified according to the cardinality of the limit set, provided $n \geq 3$.

(1) If L is empty, then M is a *spherical space form*, that is, $M = S^n/F$, where F is a finite subgroup of $SO(n+1)$.

(2) If L is a singleton, then M is an *Euclidean space form*, that is, $M = \boldsymbol{R}^n/\Gamma$, where Γ is a discrete group of Euclidean motions of $\boldsymbol{R}^n$. In this case M is known to have an n-torus as a finite covering.

(3) If L consists of two points, then M is a *Hopt manifold*, that is, $M = (\boldsymbol{R}^n - \{O\})/\Gamma$, where Γ is a discrete group of Eucledean similarities which keep O fixed. Then M has $S^{n-1} \times S^1$ as a finite covering

Theorem 5. *Suppose that Γ is contained in the isotropy subgroup of some point of S^n. Then M is elementary.*

This theorem is a slight amelioration of a theorem of Fried ([2]), which originally postulates that the image of the developing map misses the point in Theorem 5.

Using Theorem 5, it is not difficult to show the following fact.

Theorem 6. *If Γ does not contain nonabelian free subgroup, then M is elementary.*

Theorem 6 is first proved by Kamishima ([5]) using a result of Goldman ([4]), under the condition that Γ is virtually solvable. However the virtually solvability is known to be equivalent to the hypothesis of Theorem 6 in the case of groups of matrices. Our proof of Theorem 6 is more straightfoward and elementary.

4. Cantor limit set. When two manifolds M_1 and M_2 carry conformally flat structures, there is a canonical way to define a conformally flat structure on $M_1 \sharp M_2$, although it is not unique. We call this new structure the connected sum of the two structures.

If M is obtained by the connected sum of finitely many elementary conformally flat manifolds and is not itself elementary, then M is called a *C-structure*. Then the limit set is clearly a Cantor set.

Recall that a Cantor set L in S^n is called *tame*, if $h(L) \subset S^1$ for some homeomorphism h of S^n. Otherwise it is called *wild*. The limit set of a C-structure is easily shown to be tame. Conversely in dimension 3 we have:

Theorem 7. *Let $n=3$. If the limit set is a tame Cantor set, then M is a C-structure.*

Theorem 8. *Let $n=3$. There exists a conformally flat manifold whose limit set is a wild Cantor set.*

An example of a wild Cantor limit set is first obtained by Bestvina-Cooper ([1]) for an open manifold and they asked about closed manifolds. Theorem 8 answers their question.

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