62. *Generalized Hypergeometric Equations with Certain Finite Irreducible Monodromy Groups*

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In this paper we shall study the irreducibility condition for monodromy groups of generalized hypergeometric equations (say GHGE, for brevity) and determine, under a certain condition, their explicit forms when they are finite groups. Recently Beukers-Heckman [1] obtained independently the same condition ([1], Propositions 2.7 and 3.3) and determined the cases of finite monodromy groups generally by a method quite different from ours. So we shall state a remark about the latter from our standpoint.

Let us consider GHGE in the form of Okubo type (see [4]);

$$(\sharp) \qquad (tI-B)\frac{dx}{dt}=Ax,$$

where $t \in S$ (the Riemann sphere), $x={}^t(x_1, \cdots, x_n)$ is a column n-vector, I is the n by n unit matrix, B is the n by n diagonal matrix diag$(0, \cdots, 0, 1)$ and A is an n by n constant matrix;

$$A=\left(\begin{array}{ccc|c} -a_1 & & 0 & 1 \\ & \ddots & & \vdots \\ 0 & & -a_{n-1} & 1 \\ \hline b_1 & \cdots & b_{n-1} & -a_n \end{array}\right)$$

with n distinct eigenvalues $-\rho_1, -\rho_2, \cdots, -\rho_n$. Moreover we assume the following;

(A) None of the quantities a_i, a_j-a_k and $\rho_l-\rho_m$ $(i, l, m=1, \cdots, n;\ j, k=1, 2, \cdots, n-1;\ j\neq k, l\neq m)$ is an integer. Moreover each ρ_j is not a positive integer.

The equation ($\sharp$) is Fuchsian on S with three regular singular points $t=0$, 1 and ∞. From (A) there is no logarithmic solution.

Remark 1. Since ($\sharp$) is accessory parameter free, the coefficients b_i are written in terms of a_j and ρ_k (see [4], § 1). Eliminating $x_1, \cdots, x_{n-1}$ and setting $x=x_n$, we obtain

$$(b) \qquad [\delta(\delta+a_1-1)\cdots(\delta+a_{n-1}-1)-t(\delta+\rho_1)\cdots(\delta+\rho_n)]x=0,$$

where $\delta=t(d/dt)$. It is just the classical GHGE which has

$${}_nF_{n-1}\left(\begin{matrix}\rho_1, \cdots, \rho_n; t\\ a_1, \cdots, a_{n-1}\end{matrix}\right)=\sum_{k=0}^{\infty}\frac{(\rho_1)_k\cdots(\rho_n)_k}{(a_1)_k\cdots(a_{n-1})_k k!}t^k$$

as its particular solution at $t=0$, where $(\alpha)_k=\alpha(\alpha+1)\cdots(\alpha+k-1)$ (for details, see [4], § 1 and § 5).

We first remind Theorem 2 in [4] which was originally obtained in [3].

Let G be the monodromy group with respect to the specific fundamental system $X=(X_1, \cdots, X_n)$ of solutions of (♯) ([4], Theorem 1). It is a group representation of the fundamental group $\pi_1(S^*)$ ($S^*=S\backslash\{0, 1, \infty\}$) into $GL(n, C)$ generated by the *circuit matrices* $\{M_\lambda\}_{\lambda=0,1}$ around 0 and 1. Let us denote $\exp(-2\pi\sqrt{-1}\,a_j)$ and $\exp(-2\pi\sqrt{-1}\,\rho_j)$ by e_j and f_j, respectively. Then M_λ is represented as

$$(1)\qquad M_0=\left[\begin{array}{ccc|c} e_1 & & 0 & (e_1-1)p_1 \\ & \ddots & & \vdots \\ 0 & & e_{n-1} & (e_{n-1}-1)p_{n-1} \\ \hline 0 & \cdots & 0 & 1 \end{array}\right], \qquad M_1=\left[\begin{array}{ccc|c} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \\ \hline (e_n-1)q_1 & \cdots & (e_n-1)q-1 & e_n \end{array}\right].$$

Theorem 2 (Okubo-Takano). *The following relations hold:*

$$(2)\qquad p_jq_j=-\frac{\prod (e_j-f_k)}{e_j(e_j-1)(e_n-1)\prod'_{k\neq j}(e_j-e_k)} =-\frac{\prod \sin\pi(a_j-\rho_k)}{\sin\pi a_j\cdot\sin\pi a_n\cdot\prod'_{k\neq j}\sin\pi(a_j-a_k)},$$

where $\prod$ *and* $\prod'$ *are* $\prod_{k=1}^{n}$ *and* $\prod_{k=1}^{n-1}$, *respectively.*

Remark 3. K. Okubo [2] determined each connection coefficients p_j and q_j explicitly (see also [4], Theorem 3). It is sufficient for our purpose to know only Theorem 2 because of the following arguments: Let us assume $p_jq_j\neq 0$ for all j. Then, if we take q_j to any preassigned non-zero values, p_j are determined uniquely by (2). Substituting those values into (1), we obtain new matrices, say $\overline{M}_\lambda$, and non-singular diagonal matrix D determined uniquely up to a scalar multiple which satisfy $\langle\overline{M}_0, \overline{M}_1\rangle=D^{-1}GD$, where $\langle\ ,\ \rangle$ is the group generated by $\overline{M}_\lambda$ in $GL(n, C)$. Namely preassigned non-zero q_j's determine a group representation equivalent to G.

Now we state the irreducibility conditions for G which was obtained independently in [1].

Theorem 4. *G is irreducible if and only if* $e_j\neq f_k\neq 1$ *for all* $j=1, 2, \cdots, n-1$ *and* $k=1, 2, \cdots, n$, i.e., *none of the quantities* $a_j-\rho_k$ *and* ρ_k *is an integer* (cf., *assumption* (A)).

From now on we assume that G *is irreducible*. Let M_{0j} be a generalized reflection

$$M_{0j}=\begin{bmatrix} 1 & & & & 0 \\ & \ddots & & 0 & \vdots \\ & & e_j & & (e_j-1)p_j \\ & 0 & & \ddots & \vdots \\ & & & & 1 \end{bmatrix} \qquad (j=1, 2, \cdots, n-1),$$

i.e., all the diagonal elements are 1 except for the j-th element which is e_j and all off-diagonal elements are 0 except for the (j, n)-th element which is $(e_j-1)p_j$. Obviously we have $M_0=M_{01}\cdots M_{0,n-1}$. Let us denote $\langle M_{01}, \cdots, M_{0,n-1}, M_1\rangle$ by $\tilde{G}$ which contains $G=\langle M_0, M_1\rangle$ as its subgroup. From the assumption on G, $\tilde{G}$ is irreducible. If $\tilde{G}$ is finite, so is G, and $\tilde{G}$ must be a finite unitary reflection group with n reflections as its generators. Such groups were completely classified by Shephard-Todd [7]. Let us denote the number $k(1\leq k\leq 37)$ of the group in table VII in [7] by STk.

In the following our purpose is to determine all cases where $\tilde{G}$ to be finite when $n\geq 3$. For the case $n=2$ it is equivalent to determine the same cases on G which was done by H. A. Schwarz [6]. From the above arguments we obtain $a_j, \rho_j \in \boldsymbol{Q}$. The invariance of the trace of A implies $\sum_j a_j=\sum_j \rho_j$. On the other hand all M_{0j} and M_1 are written in terms of e_j and f_k. Thus we may assume, by (A) and Theorem 4,

(3) $\qquad 0<a_j,\ \rho_k<1 \qquad (j=1, 2, \cdots, n-1;\ k=1, 2, \cdots, n).$

Lemma 5. *If $\tilde{G}$ $(n\geq 3)$ is finite, then the dimension n must be 3.*

Let H be the inverse matrix of h;

$$h=\begin{bmatrix} 1 & 0 & p_1 \\ 0 & 1 & p_2 \\ q_1 & q_2 & 1 \end{bmatrix}.$$

The existenece of H follows from (3) and $a_3 \notin \boldsymbol{Z}$, for $\det h=\prod(\sin \pi\rho_j/\sin \pi a_j)$. We may assume $a_1<a_2$ and $\rho_1<\rho_2<\rho_3$.

Lemma 6. *H is taken to be hermitian by an appropriate choice of a diagonal matrix D (Remark 3) if and only if $\rho_1<a_1<\rho_2<a_2<\rho_3$.*

This condition leads $0<a_3<1$. The transformed groups of G and $\tilde{G}$ by D are written again, for simplicity, by G and $\tilde{G}$, respectively,

Lemma 7. *If H is hermitian, then it is G- and $\tilde{G}$-invariant. Moreover it is positive definite from* (3).

Noting these facts in addition to the result ([8], 3.4) due to T. A. Springer we obtain

Theorem 8. *$\tilde{G}$ is finite if and only if the set $(a_1, a_2, a_3;\ \rho_1, \rho_2, \rho_3)$ under the condition* (3) *takes one of the following values up to the complex conjugate:*

(I) $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}; \frac{1}{9}, \frac{4}{9}, \frac{7}{9}\right)$; *$\tilde{G}\simeq ST25$ and imprimitive $G\simeq(\boldsymbol{Z}/3\boldsymbol{Z}\times\boldsymbol{Z}/3\boldsymbol{Z}\times\boldsymbol{Z}/3\boldsymbol{Z})\rtimes(\boldsymbol{Z}/3\boldsymbol{Z})$.*

(II) $\left(\frac{1}{2}, \frac{2}{3}, \frac{1}{3}; \frac{1}{12}, \frac{7}{12}, \frac{5}{6}\right)$; *$G=\tilde{G}\simeq ST26$.*

(III) $\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{3}; \frac{1}{18}, \frac{7}{18}, \frac{13}{18}\right)$; *$G=\tilde{G}\simeq ST26$.*

(IV) $\left(\frac{n}{m}, \frac{1}{2}, \frac{1}{2}; \frac{n}{3m}, \frac{m+n}{3m}, \frac{2m+n}{3m}\right)$ *for any $m, n \in \boldsymbol{N}$ with $m\geq 3$, $1\leq n<m$, $(n, m)=1$; $\tilde{G}\simeq G(m, 1, 3)$ and*

(i) $G \simeq \left\langle G\left(\frac{m}{2}, 1, 3\right), \zeta_m I \right\rangle$, *where* $\zeta_m = \exp(2\pi\sqrt{-1}/m)$, *if m is even,*

(ii) $G = \tilde{G}$ *if m is odd.*

Remark 9. The following had been presumed from a differential equational point of view; *if* G *is a finite group, then so is* $\tilde{G}$. However. the groups $\tilde{G}$, for examples, corresponding to the cases on the list 8.3 in [1] are all infinite from Theorem 8.

Remark 10. We found the above case (IV) intuitively and checked that there is no other finite imprimitive $\tilde{G}$ when $m \leq 6$ by using MACSYMA on DEC VAX-11/750. The same fact for any m follows from Theorem 5.8 in [1] which is the only one result of [1] we used. Moreover, in the case (I), the natural reflection subgroup of G acts reducibly on C^3 ([1], Theorem 5.3). We also note that (II) and (III) are (5/6)-shift of No. 9 and the complex conjugate of (1/2)-shift of No. 10 on the table 8.3 in [1], respectively.

Remark 11. Finally we have to point out that the condition stated in Lemma 6 leads H to be Hermitian and that, in [1], the same implies the positive definiteness of invariant forms.

Almost all results in this paper were announced in the symposium at RIMS, October 1988 [5]. Details will appear in elsewhere.

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