61. *Limiting Amplitude Principle for Acoustic Propagators in Perturbed Stratified Fluids*

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Introduction. Recently the spectral problems for acoustic operators $L=-c(x)^2\Delta$ in perturbed stratified fluids have been studied by several authors ([1], [2], [7], [8]). Under suitable assumptions on the behavior of sound speed $c(x)$ at infinity, non-existence of eigenvalues and the principle of limiting absorption have been proved for the operator L. In the present note we study the principle of limiting amplitude for L which has not been discussed in detail in the works above.

1. Limiting amplitude principle. The precise formulation of the obtained result requires several notations and assumptions.

We work in the 3-dimensional space R^3_x and write the coordinates in R^3_x as $x=(y, z)$ with $y\in R^1$ and $z=(z_1, z_2)\in R^2$. Let Δ be the 3-dimensional Laplace operator and let $c_0(y)>0$ be the sound speed in the fluid under consideration, which depends on the depth variable y only. In particular, we here are interested in the case where $c_0(y)$ takes the constant values c_-, c_0 and c_+ for $y<0$, $0<y<h$ and $y>h$, respectively. Then the acoustic wave in the stratified fluid is governed by the wave equation $\partial^2u/\partial t^2-c_0(y)^2\Delta u=0$. On the other hand, the acoustic wave in a perturbed stratified fluid which we consider here is also governed by a similar equation $\partial^2u/\partial t^2-c(x)^2\Delta u=0$, where the sound speed $c(x)$ is assumed to satisfy the following assumptions:

(A.1) $\quad 0<c_m\leqq c(x)\leqq c_M \quad$ for some c_m and c_M.

(A.2) $\quad c(x)-c_0(y)=O(|x|^{-\rho}),\quad |x|\to\infty, \quad$ for some $\rho>1$.

We consider the above wave equation in the Hilbert space $L^2(R^3_x; c(x)^{-2}dx)$. Define the acoustic operator L as $L=-c(x)^2\Delta$. Then L is symmetric in this space and it admits a unique selfadjoint realization. We denote by the same notation L this self-adjoint realization and by $R(\zeta; L)$, $\operatorname{Im}\zeta\neq 0$, the resolvent of L; $R(\zeta; L)=(L-\zeta)^{-1}$. As is easily seen, the operator L is positive (zero is not an eigenvalue) and the domain $D(L)$ is given by $D(L)=H^2(R^3_x)$, $H^s(R^3_x)$ being the Sobolev space of order s. We here summarize the spectral properties of L obtained by the works [1], [2], [7] and [8] under assumptions (A.1) and (A.2): (i) L has no eigenvalues; (ii) The boundary values $R(\lambda\pm i0; L)$, $\lambda>0$, of $R(\lambda\pm i\kappa; L)$ as $\kappa\to 0$ exist as an operator from L^2_α into $L^2_{-\alpha}$ for $\alpha>1/2$, where $L^2_\alpha=L^2_\alpha(R^3_x)$ is the weighted L^2

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space defined by $L^2_\alpha = L^2(R^3_x; (1+|x|)^{2\alpha}dx)$; (iii) The operators $R(\lambda \pm i0; L)$: $L^2_\alpha \to L^2_{-\alpha}$, $\alpha > 1/2$, have the local Hölder continuity in $\lambda > 0$ in the uniform operator topology. Among these results, (i) has been proved by [8].

We now formulate the main theorem. Consider the Cauchy problem

$$\partial^2 u/\partial t^2 + Lu = \exp(-it\sqrt{\omega})f, \qquad \omega > 0, \tag{1}$$

with zero initial conditions $u(0, x) = (\partial u/\partial t)(0, x) = 0$. Then we obtain the following result on the asymptotic behavior as $t \to \pm\infty$ of the solution $u(t, x)$ to problem (1).

Theorem. *Let the notations be as above. Assume that $c_- \neq c_+$ and that* (A.1) *and* (A.2) *are satisfied. If f is in $L^2_\alpha(R^3_x)$ with $\alpha > 3/2$, then the solution $u(t, x)$ to* (1) *behaves like*

$$u(t, x) = \exp(-it\sqrt{\omega})R(\omega \pm i0; L)f + o(1), \qquad t \to \pm\infty,$$

strongly in $L^2_{-\alpha}(R^3_x)$.

The theorem above implies the validity of the limiting amplitude principle for the operator L.

2. Sketch of proof. The proof is based on the abstract theorem due to Eidus [3]. According to this theorem, the main theorem above follows from the two properties below of the resolvent $R(\lambda \pm i0; L)$: (R.1) local Hölder continuity in $\lambda > 0$ in the uniform operator topology from L^2_α into $L^2_{-\alpha}$ for $\alpha > 3/2$; (R.2) behavior at low frequencies

$$\|R(\lambda \pm i0; L)\|_{\alpha \to -\alpha} = O(\lambda^{-d}), \qquad \lambda \to 0,$$

for some d, $0 < d < 1/2$, where $\|\cdot\|_{\alpha \to \beta}$ denotes the operator norm from L^2_α into L^2_β. Property (R.1) has been already established. Thus it suffices to verify the property (R.2) only.

To prove this property, it is convenient to work in the usual space $L^2(R^3_x)$ rather than in the original space $L^2(R^3_x; c(x)^{-2}dx)$. Let the positive constants c_-, c_0 and c_+ be as above. We deal with only the case $c_0 < c_- < c_+$ with normalization $c_+ = 1$. A similar argument with a slight modification applies to the other cases. We also consider only the low frequency λ, $0 < \lambda \ll 1$. Set $V(x) = c(x)^{-2} - c_+^{-2} = c(x)^{-2} - 1$ and define the self-adjoint operator $K(\lambda)$ acting on the space L^2 by $K(\lambda) = -\Delta - \lambda V$. Then we have

$$R(\lambda \pm i\kappa; L) = Q(\lambda \pm i\kappa; K(\lambda))c(x)^{-2},$$

where $Q(\lambda \pm i\kappa; K(\lambda)) = (K(\lambda) - (\lambda \pm i\kappa c(x)^{-2}))^{-1}$. We assert that for $\alpha > 3/2$

$$\|Q(\lambda \pm i\kappa; K(\lambda))\|_{\alpha \to -\alpha} = O(|\log \lambda|) \tag{2}$$

as $\lambda \to 0$ uniformly in κ, $0 < \kappa \leqq 1$. This implies (R.2) immediately. The proof of (2) is done by making use of the commutator method due to Mourre [5].

We consider only the + case. Let $\chi(x) \in C_0^\infty(R^3_x)$, $0 \leqq \chi \leqq 1$, be a smooth cut-off function such that χ has support in $\{x : |x| < 2\}$ and $\chi = 1$ on $|x| < 1$. For $\varepsilon > 0$ small enough, we set

$$V_\varepsilon(x) = V_0(y) + \chi(\varepsilon x)(c(x)^{-2} - c_0(y)^{-2})$$

with $V_0(y) = c_0(y)^{-2} - c_+^{-2}$, so that $V_\varepsilon(x) = V(x)$ for $|x| < \varepsilon^{-1}$. We also define the self-adjoint operator $K(\varepsilon; \lambda)$ as $K(\varepsilon; \lambda) = -\Delta - \lambda V_\varepsilon$. Let $A = (-i/2)(x \cdot \nabla_x + \nabla_x \cdot x)$ be the generator of the dilation unitary group. Then we have

Lemma. *Let the notations be as above. Let* $f_\lambda(s)\in C_0^\infty(R_s^1)$, $0\leqq f_\lambda\leqq 1$, *be a function such that* f_λ *has support in* $(\lambda/3, 3\lambda)$ *and* $f_\lambda=1$ *on* $[\lambda/2, 2\lambda]$. *Then, for* ε, $0<\varepsilon\leqq\varepsilon_0$, *small enough, the commutator* $B(\varepsilon;\lambda)=[K(\varepsilon;\lambda), A]$ *satisfies*

$$M(\varepsilon;\lambda)=if_\lambda(K(\lambda))B(\varepsilon;\lambda)f_\lambda(K(\lambda))\geqq\gamma\lambda f_\lambda(K(\lambda))^2$$

in the form sense, where $\gamma>0$ *is independent of* λ, $0<\lambda\ll 1$.

The lemma above plays a central role in proving the resolvent estimate (2) at low frequencies by use of the commutator method and this is proved by constructing explicitly the Green function of the ordinary differential operator $-d^2/dy^2-\lambda V_0(y)$.

Remark. The original Mourre commutator method is applied to $K(\lambda)$ rather than to $K(\varepsilon,\lambda)$. This requires the assumption $c(x)-c_0(y)=O(|x|^{-2})$, $|x|\to\infty$. However, we can dispense with such a restrictive assumption by introducing an ε-dependent cut-off on the coefficient $V(x)$ (see Tamura [6]).

The above lemma enables us to define

$$G_\kappa(\varepsilon;\lambda)=(K(\lambda)-\lambda-i\kappa c(x)^{-2}-i\varepsilon M(\varepsilon;\lambda))^{-1}: L^2\to L^2$$

for κ, $0<\kappa\leqq 1$ and ε, $0<\varepsilon\leqq\varepsilon_0$. By the differential inequality technique initiated by Mourre, we can prove that for $\alpha>3/2$

$$\|(d/d\varepsilon)G_\kappa\|_{\alpha\to-\alpha}\leqq C(1+\varepsilon^{-1/2}\|G_\kappa\|_{\alpha\to-\alpha}^{1/2}+\varepsilon^{\rho-2}\|G_\kappa\|_{\alpha\to-\alpha}),$$

ρ, $\rho>1$, being as in (A.2), and

$$\|G_\kappa(\varepsilon_0;\lambda)\|_{\alpha\to-\alpha}=O(|\log\lambda|),\qquad \lambda\to 0,$$

uniformly in κ. Then it follows that

$$\|G_\kappa(\varepsilon;\lambda)\|_{\alpha\to-\alpha}=O(|\log\lambda|),\qquad \lambda\to 0,$$

uniformly in κ and ε. This proves the assertion (2).

3. Remark. We shall explain briefly the reason why the additional assumption $c_-\neq c_+$ is assumed in the statement of main theorem. If $0<c_0<c_-=c_+$, then the ordinary differential operator $-d^2/dy^2-\lambda V_0(y)$ with $V_0(y)=c_0(y)^{-2}-c_+^{-2}$ has at least one negative eigenvalue for any $\lambda>0$ small enough. This makes it difficult to prove the resolvent estimate (R.2) at low frequencies and more elaborate analysis seems to be required to overcome such a difficulty. This is the reason why we do not consider the case $c_-=c_+$ here.

The details will be discussed in Kikuchi-Tamura [4].

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