59. *Recurrent Fuchsian Groups whose Riemann Surfaces have Infinite Dimensional Spaces of Bounded Harmonic Functions*

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§1. **Introduction and statement of results.** A Fuchsian group Γ acts on both the unit disc $\boldsymbol{D}$ and on the unit $\boldsymbol{S}^1$. Such a group is said to be recurrent if, for any positive measure subset $A \subset \boldsymbol{S}^1$, $\sharp\{\gamma \in \Gamma : m(A \cap \gamma A) > 0\} = \infty$. Such groups have been considered as a subject of study in their own right primarily since the appearance of Dennis Sullivan's profound paper [9].

The function theory corresponding to such groups is not yet understood. For example, if $\mathfrak{R} = \boldsymbol{D}/\Gamma$ (we will use this notation throughout this note), the structure of the spaces of bounded harmonic or bounded holomorphic functions are not yet clear. Taniguchi constructed examples of Fuchsian groups such that the space of bounded harmonic functions on $\mathfrak{R}$, $HB(\mathfrak{R})$, is finite dimensional [11].

Now bounded harmonic functions on $\mathfrak{R}$ arise from integrating Γ-invariant measurable functions on $\boldsymbol{S}^1$ against the Poisson kernel and projecting down to $\mathfrak{R}$ from $\boldsymbol{D}$. We define harmonic measure class on $\mathfrak{R}$ to be the σ-algebra of Γ-invariant measurable subsets of $\boldsymbol{S}^1$, with the measure m_p on $\boldsymbol{S}^1$ associated to a point $p \in \boldsymbol{D}$ just the visual measure from p with respect to the hyperbolic metric on $\boldsymbol{D}$.

It is easy to see that the notions of positive measure and zero measure sets are well-determined in this measure class (although the measure of a positive measure set is only defined if it is 1 or 0). Furthermore, the notion of an atom in this harmonic measure class is well-defined as an ergodic component with positive measure. Given the definition of O_{HB}^{∞} from [2, pp. 119–128], one easily deduces.

Lemma 1. *$\mathfrak{R} \in O_{HB}^{\infty}$ if and only if Γ decomposes $\boldsymbol{S}^1$, up to measure zero, into a union of positive measure ergodic components for its action.*

The proof is left to the reader. Given Lemma 1 and his examples, Taniguchi proffered

Conjecture [1, p. 4]. *If Γ is a recurrent Fuchsian group then the Riemann surface $\mathfrak{R} = \boldsymbol{D}/\Gamma$ is in O_{HB}^{∞}.*

This note concern two points. The first is that the conjecture is false. Let K denote the usual middle thirds Cantor set, and $\mathfrak{R}_K$ denote the Riemann

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surface $\hat{C}\backslash K$. Let Γ_K denote the Fuchsian group such that $\boldsymbol{D}/\Gamma_K=\mathcal{R}_K$. We will sketch in §2 a demonstration of

Theorem 1. *The Riemann surface $\hat{C}\backslash K$ is not in O_{HB}^{∞}, while its Fuchsian group Γ_K is recurrent.*

The second point is the following: $\mathcal{R}$ is said to be in O_{HB}^{n} if and only if up to measure zero the action of Γ splits $\boldsymbol{S}^1$ into at most n disjoint positive measure ergodic components. Taniguchi constructed surfaces in $O_{HB}^{n}\backslash O_{HB}^{n-1}$ (i.e. the group action has precisely n ergodic components). This construction generalizes after but an observation to generate straightforward examples in $O_{HB}^{\infty}\backslash\bigcup_{n\in\boldsymbol{Z}_+}O_{HB}^{n}$. Compare with the example in [3, §24]. This is presented in §3. The examples are

Theorem 2. *If $\mathcal{R}$ is a finite volume Riemann surface and $\mathcal{R}'$ is an abelian cover of $\mathcal{R}$ of rank at least* 3, *let $\mathcal{R}''$ be a $\boldsymbol{Z}$-cover of $\mathcal{R}'$ corresponding to a simple closed curve $\gamma\subset\mathcal{R}'$. Then $\mathcal{R}''\in O_{HB}^{\infty}\backslash\bigcup_{n\in\boldsymbol{Z}_+}O_{HB}^{n}$.*

I had a long conversation with Peter Jones regarding this material, from which I learned most of the classical analysis associated with the problem. Dennis Sullivan pestered me constantly to make sure I knew what I was talking about, aside from being a source of great inspiration and encouragement. And of course my Japanese hosts at Kyoto University are most humbly thanked for their generosity during my visit. Finally, I'd like to thank Masahiko Taniguchi for reading a preliminary version of this manuscript, suggesting changes, and finding some of the references for me.

§2. Theorem 1. *A sketch of the proof.* The theorem will follow naturally as a consequence of several lemmas. Each of these lemmas is well-known and hence only references are given.

Lemma 2. *If $C\subset\boldsymbol{C}$ is compact, and if the Riemann surface $\hat{C}\backslash C$ has a Green's function, then it has no atoms in harmonic measure class.*

This is an immediate consequence of [2, Folgesatz 11.9] or [10, Theorem III.5F].

Let $\mathcal{F}$ be the Dirichlet fundamental domain for Γ, and $\mathrm{cl}(\mathcal{F})$ the closure of $\mathcal{F}$ as a subset of $\boldsymbol{C}$. Let $\mathcal{F}^*=\mathrm{cl}(\mathcal{F})\cap\boldsymbol{S}^1$. Sullivan established

Lemma 3 [9, Theorem 4, p. 488]. *Γ is recurrent if and only if the Lebesgue measure of $\mathcal{F}^*$ in $\boldsymbol{S}^1$ is* 0.

Now let $C\subset\boldsymbol{R}$ be compact, containing at least two points. The domain $\mathcal{R}_C=\hat{\boldsymbol{C}}\backslash C$ is called a Denjoy domain. Let Γ_C be the Fuchsian group such that $\mathcal{R}_C=\boldsymbol{D}/\Gamma_C$, and $\mathcal{F}_C$, $\mathcal{F}_C^*$ as above. We immediately have

Lemma 4 (due to Beurling (see [8, §4] or [7])). *$\mathcal{F}_C^*$ has null angular measure if and only if the linear measure of C is null.*

Combining these three lemmas one sees

Theorem 1′. *If $C\subset\boldsymbol{R}$ is a compact null set such that $\mathcal{R}_C$ has a Green's function then Γ_C is recurrent and $\mathcal{R}_C$ has no atoms for harmonic measure class and is thus not in O_{HB}^{∞}.*

Our desired result is an immediate corollary as the fact that $\hat{C}\backslash K$

supports a Green's function is standard [12, ch. III, §4].

§3. Construction of surfaces in $O_{HB}^{\infty}\setminus\bigcup_{n\in\boldsymbol{Z}_+}O_{HB}^{n}$. Lyons and Sullivan [5] extend a result of Mori [6] to show that abelian covers of rank at least three of closed Riemann surfaces carry Green's functions but have no positive nonconstant harmonic functions. Hence the space of bounded harmonic functions on such a Riemann surface is trivial, i.e. just the constants. These furnish an abundance of examples of recurrent groups in $O_{HB}^{1}\setminus O_G$ (again using the notation of [2], [3]). By considering finite covers of such surfaces, Taniguchi [11] constructed his examples of surfaces in $O_{HB}^{n}\setminus O_{HB}^{n-1}$ for all $n\geq 2$.

In the same fashion we now generate surfaces in $O_{HB}^{\infty}\setminus\bigcup_{n\in\boldsymbol{Z}_+}O_{HB}^{n}$. Let $\mathcal{R}_{g,n}$ be a hyperbolic Riemann surface of finite volume, with $2g+n\geq 4$ (we drop the subscripts for convenience). Then $\mathcal{R}=\boldsymbol{D}/\Gamma$. We will work strictly with homology covers as this is notationally a bit easier. The homology cover of $\mathcal{R}$ is $\mathcal{R}'=\boldsymbol{D}/[\Gamma,\Gamma]$, so that $\mathcal{R}'$ is a regular cover of $\mathcal{R}$ with group $\Gamma'=\Gamma/[\Gamma,\Gamma]$. Thus $\mathcal{R}=\mathcal{R}'/\Gamma'$. Our surfaces are $\boldsymbol{Z}$-covers of such $\mathcal{R}'$.

Take a countable number of copies of this homology cover, $\mathcal{R}'_n$ indexed by $\boldsymbol{Z}$, and break each open along their copies of a simple closed geodesic γ. Label the two sides of this cut on $\mathcal{R}'_n$ by γ_n^+ and γ_n^-. Now glue γ_n^+ to γ_{n+1}^- for each $n\in\boldsymbol{Z}$, thus obtaining a $\boldsymbol{Z}$-cover of $\mathcal{R}'$. This surface is our desired $\mathcal{R}''$.

Theorem 2. $\mathcal{R}''\in O_{HB}^{\infty}\setminus\bigcup_{n\in\boldsymbol{Z}_+}O_{HB}^{n}$.

To see that $\mathcal{R}''\in O_{HB}^{\infty}\setminus\bigcup_{n\in\boldsymbol{Z}_+}O_{HB}^{n}$, we pick a basepoint $p''\in\mathcal{R}''$ and consider the unit tangent circle $T_{p''}^{1}\mathcal{R}''$ to $\mathcal{R}''$ at p''. We consider this as the space parametrizing directed geodesics through p'', with γ_x being the geodesic through p'' in direction $x\in T_{p''}^{1}\mathcal{R}''$. Let

$$\Theta_{p''}=\{x\in T_{p''}^{1}\mathcal{R}'' : \gamma_x \text{ eventually leaves any compact subset of } \mathcal{R}''\}.$$

It is well known [4] that $\mathcal{R}''$ has a Green's function if and only if $m_\theta(\Theta_{p''})=1$. This is clearly the case, as $\mathcal{R}''$ covers $\mathcal{R}'$ which has Green's function.

But more importantly we see that for almost all x the geodesic γ_x is eventually in $\mathcal{R}'_n$ for some n. This is because on projection down to $\mathcal{R}'$ almost all geodesics intersect γ only finitely often. Thus all harmonic measure on $\mathcal{R}''$ comes from the contributions at infinity of each $\mathcal{R}'_n$. That each of these is atomic, hence completing the arguement, is evident by group invariance.

References

[1] Agard, S.: Mostow rigidity on the line. A survey. Holomorphic Functions and Moduli II, MSRI **11**, 1–12 (1988).

[2] Constantinescu, C. and Cornea, A.: Ideale Ränder Riemannischer Fläschen. Springer-Verlag, Berlin (1963).

[3] ——: Über den idealen Rand und einige seiner Anwendungen bei der Klassifi-

kation der Riemannschen Fläschen. Nagoya Math. J., **13**, 169–233 (1958).
[4] Hopf, E.: Ergodic theory and the geodesic flow on surfaces of constant negative curvature. Bull. Amer. Math. Soc., **77**, 863–887 (1971).
[5] Lyons, T. and Sullivan, D.: Function theory, random paths and covering spaces. J. Diff. Geom., **19**, 299–323 (1984).
[6] Mori, A.: A note on unramified abelian covering surfaces of a closed Riemann surface. J. Math. Soc. Japan, **6**, 162–176 (1954).
[7] Pommerenke, C.: On Fuchsian groups of accessible type. Ann. Acad. Sci. Fenn., ser. A. I. **7**, 249–258 (1982).
[8] Rubel, L. and Ryff, J.: The bounded weak-star topology and the bounded analytic functions. J. Func. Anal., **5**, 167–183 (1970).
[9] Sullivan, D.: On the ergodic theory at infinity of an arbitrary group of hyperbolic motions. Riemann surfaces and related topics. Proceedings of the 1978 Stony Brook Conference, Ann. of Math. Studies, **97**, 465–496 (1981).
[10] Sario, L. and Nakai, M.: Classification theory of Riemann surfaces. Springer-Verlag, Berlin (1970).
[11] Taniguchi, M.: Examples of discrete groups of motions conservative but not ergodic at infinity. Ergod. Th. and Dynam. Sys., **8**, 633–636 (1988).
[12] Tsuji, M.: Potential Theory in Modern Function Theory. Maruzen, Tokyo (1959).