58. *Uniqueness and Existence of Viscosity Solutions of Generalized Mean Curvature Flow Equations*

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1. **Introduction.** We construct unique global continuous viscosity solutions of the initial value problem in $\boldsymbol{R}^n$ for a class of degenerate parabolic equations that we shall call *geometric*. A typical example is

$$(1) \qquad u_t-|\nabla u|\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)-\nu|\nabla u|=0 \qquad \left(u_t=\frac{\partial u}{\partial t},\ \nabla u=\operatorname{grad} u,\ \nu\in\boldsymbol{R}\right).$$

Our method is based on the comparison principle of viscosity solutions developed recently by Jensen [8] and Ishii [6]. However, as is observed from (1), our equation is singular at $\nabla u=0$ so we are forced to extend their theory to our situation.

The equation (1) has a geometric significance because γ-level surface $\Gamma(t)$ of u moves by its mean curvature when $\nu=0$ provided that ∇u does not vanish on $\Gamma(t)$. Such a motion of surfaces has been studied by many authors [1–5]. However, so far whole *unique* evolution families of surfaces were only constructed under geometric restrictions on initial surfaces such as convexity [3, 5] except $n=2$ [1, 4]. When $n=2$, Grayson [4] has shown that any embedded curve moved by its curvature never becomes singular unless it shrinks to a point. However when $n\geq 3$ even embedded surfaces may become singular before it shrinks to a point.

Our goal is to construct whole evolution family of surfaces even after the time when there appear singularities. This program is carried out by Angenent [1] when $n=2$. Contrary to [1] we avoid parametrization and rather understand surfaces as level sets of viscosity solutions of (1). Let $D(t)$ denote the open set of $x\in\boldsymbol{R}^n$ such that $u(x,t)>\gamma$. When the equation is geometric, it turns out that the family $(\Gamma(t), D(t))$ $(t\geq 0)$ is uniquely determined by $(\Gamma(0), D(0))$ and is independent of u and γ. By unique existence of viscosity solution of (1) we have a unique family of $(\Gamma(t), D(t))$ for all $t\geq 0$ provided $D(0)$ is bounded open and that $\Gamma(0)$ $(\subset\boldsymbol{R}^n\backslash D(0))$ is compact. As is expected, we conclude that $(\Gamma(t), D(t))$ becomes empty in a finite time provided $\nu\leq 0$. This extends a result of Huisken [5] where he proved that $\Gamma(t)$ disappears in a finite time provided $\Gamma(0)$ is a uniformly convex C^2 hypersurface.

In this note we state our main results almost without proofs; the details will be published elsewhere.

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2. A parabolic comparison principle. For $h: L \to \boldsymbol{R}$ ($L \subset \boldsymbol{R}^d$) we associate its *lower (upper) semicontinuous relaxation* $h_*(h^*): \bar{L} \to \bar{\boldsymbol{R}} = \boldsymbol{R} \cup \{\pm\infty\}$ defined by

$$h_*(z) = \lim_{\varepsilon \downarrow 0} \inf_{|z-y|<\varepsilon} h(y), \quad h^*(z) = -(-h)_*(z), \quad z \in \bar{L}.$$

Let Ω be an open subset of $\boldsymbol{R}^n$. We write $J(\Omega) = \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n \times S^{n\times n}$ and $W = \Omega \times \boldsymbol{R} \times (\boldsymbol{R}^n \backslash \{0\}) \times S^{n\times n}$ where $S^{n\times n}$ denotes the space of $n\times n$ real symmetric matrices. Let $F = F(t, x, s, p, X)$ be a real valued function defined in $(0, T] \times W$ for $T<\infty$. Since W is dense in $J(\Omega)$, we see F^*, F_*: $[0, T] \times J(\Omega) \to \bar{\boldsymbol{R}}$. Any function u: $\Omega_T \to \boldsymbol{R}$ is called a *viscosity sub-(super) solution* of

$$u_t + F(t, x, u, \nabla u, \nabla^2 u) = 0 \quad \text{in} \quad \Omega_T = (0, T] \times \Omega \tag{2}$$

if $u^* < \infty (-\infty < u_*)$ on $\bar{\Omega}_T$ and if, whenever $\phi \in C^2(\Omega_T)$, $(t, y) \in \Omega_T$ and $(u^* - \phi)(t, y) = \max_{\Omega_T} (u^* - \phi)$ $((u_* - \phi)(t, y) = \min_{\Omega_T} (u_* - \phi))$

$$\phi_t(t, y) + F_*(t, y, u^*(t, y), \nabla\phi(t, y), \nabla^2\phi(t, y)) \le 0$$
$$(\phi_t(t, y) + F^*(t, y, u_*(t, y), \nabla\phi(t, y), \nabla^2\phi(t, y)) \ge 0).$$

If u: $\Omega_T \to \boldsymbol{R}$ is both a viscosity sub- and supersolution of (2), u is called a *viscosity solution* of (2). We say F is *degenerate elliptic* if

$$F(t, x, s, p, X+Y) \le F(t, x, s, p, X)$$

for every $(t, x, s, p, X) \in W$ and $Y \ge O$. We say (2) is a *degenerate parabolic* equation if F is degenerate elliptic.

Example 1. The equation (1) is degenerate parabolic since (1) is expressed in the form (2) by taking

$$F(t, x, s, p, X) = -\text{trace}\,((I - \bar{p}\,{}^t\bar{p})X) - \nu|p|, \qquad \bar{p} = p/|p|.$$

Example 2. For $\omega \ge 0$ we set

$$\psi^{\pm}(t, x) = \mp(|x| - \omega t)^4 \quad \text{if } |x| > \omega t, \text{ otherwise } \psi^{\pm}(t, x) = 0. \tag{3}$$

Suppose that F is elliptic and satisfies

$$-\nu|p| \le F(t, x, s, p, O) (\le \mu|p|) \quad \text{in } W \tag{4}$$

for some constant $\nu \geqq 0 (\mu \ge 0)$. Then $\psi^+(\psi^-)$ is a viscosity super-(sub) solution of (2) with $\Omega = \boldsymbol{R}^n$ provided $\omega \ge \nu$ $(\omega \ge \mu)$.

Example 3. The function $U_{\xi h}(t, x) = h(2(n-1)t + |x-\xi|^2)$ is a viscosity solution of (1) in $\boldsymbol{R}^n_T$ for every T when $\nu = 0$ provided that h is a continuous monotone function on $\boldsymbol{R}$.

We now state our main comparison result.

Theorem 4. *Let Ω be bounded and let F: $(0, T] \times W \to \boldsymbol{R}$ is continuous, degenerate elliptic and independent of $x \in \Omega$. Assume that there is a constant $c = c(\Omega, T, M, n)$ such that the function $s \mapsto F(t, s, p, X) + cs$ is nondecreasing in $s \in \boldsymbol{R}$ for all $t \in (0, T]$, $|s| \le M$, $p \in \boldsymbol{R}^n \backslash \{0\}$, $X \in S^{n\times n}$. Suppose furthermore that*

$$-\infty < F_*(t, s, 0, O) = F^*(t, s, 0, O) < \infty, \quad t \in (0, T], \quad s \in \boldsymbol{R}. \tag{5}$$

Let u and v be, respectively, viscosity sub- and supersolutions of (2) in Ω_T. If $u^* \le v_*$ on $\partial_p \Omega_T = \{0\} \times \Omega \cup [0, T] \times \partial\Omega$, then $u^* \le v_*$ on Ω_T.

Remark 5. If two inequalities in (4) hold for F and $(t, s, X) \mapsto F(t, s, p, X)$ is equicontinuous for small p, then (5) holds. In particular Theorem 4 is applicable to the equation (1). Although our proof is based on a parabolic version of Ishii's Proposition 5.1 in [6] (cf. [7]), new idea is necessary to prove Theorem 4 since F is not continuous at $p=0$ even if we consider its elliptic version. We note that Theorem 3.1 in [6] can be extended even if F is not continuous at $p=0$ provided (5) holds. Using Perron's method as in [6] we obtain an existence result.

Theorem 6. *Let Ω and F be as above. Suppose that there is a viscosity subsolution f and a viscosity supersolution g of* (2) *such that f, g are locally bounded in $\bar{\Omega}_T$, $f \leq g$ in Ω_T and $f_* = g^*$ on $\partial_p \Omega_T$. Then there is a viscosity soluion u of* (2) *satisfying $u \in C(\bar{\Omega}_T)$ and $f \leq u \leq g$ on $\bar{\Omega}_T$, where $\bar{\Omega}_T = \partial_p \Omega_T \cup \Omega_T$.*

3. Geometric equations. We consider a special class of degenerate parabolic equations including (1).

Definition 7. A function $F: (0, T] \times W \to \boldsymbol{R}$ is called *geometric* if F does not depend on $s \in \boldsymbol{R}$ i.e.

$$F(t, x, s, p, X) = F(t, x, p, X)$$

and for every $\lambda > 0$ and $\sigma \in \boldsymbol{R}$ it holds

$$F(t, x, \lambda p, \lambda X + \sigma p\, {}^t p) = \lambda F(t, x, p, X).$$

Theorem 8. *Suppose that F is degenerate elliptic and geometric in $(0, T] \times W$. If u is a locally bounded viscosity sub-(super) solution of* (2) *in Ω_T, so is $\theta(u)$ whenever $\theta: \boldsymbol{R} \to \boldsymbol{R}$ is a continuous nondecreasing function.*

The proof depends on approximation of u by semiconvex Lipschitz functions. Example 3 follows from Theorem 8.

4. Evolutions of level surfaces. Suppose that $a \in C(\boldsymbol{R}^n)$ and $a - \alpha$ is compactly supported for some $\alpha \in \boldsymbol{R}$. Let u_a denote a viscosity solution of (2) in Ω_T such that $u_a \in C(\bar{\Omega}_T)$ with $u(0, x) = a(x)$ and that $u - \alpha$ has a compact support in $\bar{\Omega}_T$. We state our uniqueness result when $\Omega = \boldsymbol{R}^n$.

Theorem 9. *For $\Omega = \boldsymbol{R}^n$ we assume F is continuous, geometric, degenerate elliptic and is independent of x in $(0, T] \times W$. Suppose that F satisfies* (4) *and* (5). *Then there is at most one viscosity solution u_a of* (2) *in Ω_T with initial data a. Moreover, if $b \geq a$ then $u_b \geq u_a$ on $\bar{\Omega}_T$.*

Proof. We may assume $\alpha = 0$. For $\psi^{\pm}$ in (3) we set

$$f_R = \min(\psi^- - R^4, 0), \qquad g_R = \max(\psi^+ + R^4, 0) \tag{6}$$

where $\omega \geq \max(\nu, \mu)$ and $R > 0$. We take R large enough so that $f_R \leq a(x) \leq b(x) \leq g_R$ holds at $t=0$. Example 2 and Theorem 8 imply that f_R and g_R are, respectively, viscosity sub- and supersolutions of (2) in $\boldsymbol{R}^n_T$. Take R_1 such that u_a, u_b, f_R, g_R are supported in $[0, T] \times B(R_1)$ where $B(R_1)$ denotes the open ball of radius R_1 centered at the origin. Applying comparison Theorem 4 with $\Omega = B(R_1)$ yields $u_b \geq u_a$. This implies uniqueness of u_a.

Theorems 8 and 9 yield:

Theorem 10. *Suppose F and u_a are as in Theorem 9. Let $\Gamma(t)$ be γ-level set of $u_a(t, \cdot)$ and $D(t)$ be a set of $x \in \boldsymbol{R}^n$ such that $u_a(t, x) > \gamma$. If $\gamma > \alpha$ then $(\Gamma(t), D(t))$ $(t \geq 0)$ is uniquely determined by $(\Gamma(0), D(0))$ and is independent of a, α and γ. We call $(\Gamma(t), D(t))$ is a solution family of* (2) *with intial data $(\Gamma(0), D(0))$.*

When (2) is quasilinear, one can construct a global viscosity solution u_a for a given initial data a.

Theorem 11. *Suppose that F and a are as in Theorem 9 and that F is linear in X. Then the viscosity solution u_a of (2) in Theorem 9 (uniquely) exists for every $T>0$.*

For general F we approximate (2) by uniformly parabolic equations and prove convergence of approximate solutions at least for $a \in C^2$. Here f_R and g_R in (6) play a role of "barriers" to get uniform estimates for first derivatives of approximate solutions. As their limit we obtain the viscosity solution u_a. For continuous a, we can approximate it by regular functions and find that Theorem 6 is applicable to get the solution.

For the equation (1) with $\nu=0$ one can construct u_a via Theorem 6 without using approximate equations. There are viscosity sub- and supersolutions f, g satisfying assumptions of Theorem 6 with $f=g=a$ at $t=0$. Indeed, for $\xi \in \boldsymbol{R}^n$ there is a decreasing continuous function h such that $U_{\xi h}(0, x) \leq a(x)$ and $U_{\xi h}(0, \xi)=a(\xi)$ where $U_{\xi h}$ is in Example 3. We define f as the supremum of such $U_{\xi h}$ and find that $f=a$ at $t=0$ and f is a viscosity subsolution of (2) in $\boldsymbol{R}^n_T$. The function g can be constructed similarly. By comparison with $f_R+\alpha$, $g_R+\alpha$ in (6), we see $f=g=\alpha$ outside $[0, T]\times B(R)$ if R is sufficiently large. Theorem 6 with $\Omega=B(R)$ yields the desired solution u_a by defining its value as α outside $B(R)$.

Corollary 12. (i) *Suppose F is as in Theorem 11. Suppose that $D(0)$ is bounded open and $\partial D(0) \subset \Gamma(0) (\subset \boldsymbol{R}^n \backslash D(0))$ is compact. Then there is a unique solution family $(\Gamma(t), D(t))$ for all $t \geq 0$ with initial data $(\Gamma(0), D(0))$.*

(ii) *Let $(\Gamma(t), D(t))$ be a solution family of (1) with $\nu \leq 0$ such that $D(0) \cup \Gamma(0)$ is bound. Then $(\Gamma(t), D(t))$ becomes empty in finite time.*

We note (i) follows from Theorems 10 and 11 with a suitable choice of a. For mean curvature flow equation (1), Example 3 yields (ii) by a comparison.

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