57. On the Deift-Trubowitz Trace Formula for the 1-dimensional Schrödinger Operator

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1. Introduction. The purpose of the present work is to prove the Deift-Trubowitz trace formula

(1) $$2i\pi^{-1}\int_{-\infty}^{\infty}\xi r_{\pm}(\xi\,;u)f_{\pm}(x,\xi\,;u)^2 d\xi=u(x)$$

for the 1-dimensional Schrödinger operator $H(u)=-\partial^2+u(x)$ with $u(x)\in\Pi_0$ such that $u', u''\in L^1(\boldsymbol{R})$, $H(u)$ has no bound states and satisfies the following conditions (A), (B) and (C):

(A) $r_{\pm}(\xi\,;u)=1+i\alpha_{\pm}\xi+o(\xi)$ as $\xi\to 0$ for some $\alpha_{\pm}\in\boldsymbol{R}$.

(B) $R_{\pm}(x)$, the Fourier transforms of $r_{\pm}(\xi\,;u)$, are absolutely continuous, and

$$\pm\int_{\alpha}^{\pm\infty}(1+x^2)|R'_{\pm}(x)|dx<\infty \qquad \text{for all } \alpha\in\boldsymbol{R}.$$

(C) $S_{+}(u)\cup S_{-}(u)\neq\varnothing$.

The notations used in the above are as follows:

$$\Pi_k=\{u\,|\,\text{real, continuous, }\lim_{|x|\to\infty}u(x)=0,\text{ and }|x|^k u(x)\in L^1(\boldsymbol{R})\},\qquad k\in[0,\infty),$$

$f_{\pm}(x,\xi\,;u)$ are the Jost solutions for $H(u)$, i.e., those solutions of

(2) $$H(u)f=-f''+u(x)f=\xi^2 f,\qquad \xi\in\boldsymbol{R}\setminus\{0\}$$

which behave like $\exp(\pm i\xi x)$ as $x\to\pm\infty$ respectively, $r_{\pm}(\xi\,;u)$ are the reflection coefficients of $H(u)$, and $S_{\pm}(u)$ are the sets of solutions $f(x)$ of (2) for $\xi=0$ such that $\lim_{x\to\pm\infty}f(x)$ exist and belong to $(0,\infty)$, respectively. Refer [2] and [3] for detail of the scattering theory of $H(u)$ with $u\in\Pi_1$ and $u\in\Pi_0$ respectively.

The trace formula (1) was first proved by Deift and Trubowitz in [2] for the potential $u(x)$ in Π_1 with $u', u''\in L^1(\boldsymbol{R})$ such that $H(u)$ has no bound states. See also [1]. Our aim is to extend the formula (1) to the potential mentioned above.

2. Darboux transformation. Let $P(H(u))$ be the set of positive solutions of the equation (2) for $\xi=0$. Suppose $P(H(u))\neq\varnothing$. Put $A_g=g^{-1}\partial g$ for $g\in P(H(u))$. Then $H(u)=A_g A_g^*$ follows, where A_g^* is the formal adjoint of A_g. We call $H^*(u\,;g)=A_g^* A_g$ the Darboux transformation of $H(u)$ by $g(x)$. Put

$$u^*(x\,;g)=u(x)-2(\log g(x))'',$$

then $H^*(u\,;g)=-\partial^2+u^*(x\,;g)$ follows.

Let $\Lambda^{(k)}$, $k\geqq 2$, be the set of potentials $u(x)\in\Pi_k$ such that $H(u)$ has no bound states, and $r_{\pm}(0;u)=-1$. The following is shown in [3] and [4].

Theorem 1. *If $v(x)\in\Lambda^{(2)}$, then $v^*(x;f_{\pm})$ belong to $\Pi_0\backslash\Pi_1$, $H^*(v;f_{\pm})$ have no bound states, and $S_{\pm}(v^*(x;f_{\pm}))\neq\varnothing$, respectively, where $f_{\pm}=f_{\pm}(x,0;v)\in P(H(v))$. Moreover*

$$r_{\sigma}(\xi;v^*(\cdot;f_{\pm}))=-r_{\sigma}(\xi;v), \tag{3}$$

$$f_{\pm}(x,\xi;v^*(\cdot;f_{\sigma}))=\pm i\xi^{-1}A^*_{f_{\sigma}}f_{\pm}(x,\xi;v) \tag{4}$$

are valid respectively ($\sigma=\pm$). Conversely, if $u(x)\in\Pi_0\backslash\Pi_1$ satisfies the conditions (A) and (B), and $S_{\pm}(u)\neq\varnothing$, then there uniquely exists $v(x)\in\Lambda^{(2)}$ such that $u(x)=v^*(x;f_{\pm})$, respectively.

Here we define $\Gamma^{(k)}_{\pm}$, the subsets of potentials in $\Pi_0\backslash\Pi_1$, by

$$\Gamma^{(k)}_{\pm}=\{u(x)\,|\,u(x)=v^*(x;f_{\pm})\text{ for }v(x)\in\Lambda^{(k)}\},\qquad k\geqq 2,$$

respectively, where $f_{\pm}=f_{\pm}(x,0;v)$. Theorem 1 implies that the Darboux transformations by $f_{\pm}$ give rise to the bijections from $\Lambda^{(k)}$ onto $\Gamma^{(k)}_{\pm}$, respectively. This enables us to characterize $\Gamma^{(k)}_{\pm}$ in terms of scattering data with the condition (C) (cf. [4]).

3. Trace formula. In [1] and [2], in addition to (1), they proved the following formulas for $u(x)\in\Lambda^{(2)}$ with $u',u''\in L^1(\boldsymbol{R})$:

$$i\pi^{-1}\int_{-\infty}^{\infty}\xi^{-1}r_{\pm}(\xi;u)f_{\pm}(x,\xi;u)^2d\xi=1, \tag{5}$$

$$-2i\pi^{-1}\int_{-\infty}^{\infty}\xi^{-1}r_{\pm}(\xi;u)f'_{\pm}(x,\xi;u)^2d\xi=u(x). \tag{6}$$

The integrals in (5) and (6) are interpreted as principal values.

Now suppose $u(x)\in\Gamma^{(2)}_{+}$. By Theorem 1, there uniquely exists $v(x)\in\Lambda^{(2)}$ such that $u(x)=v^*(x;f_{-})$, where $f_{\pm}=f_{\pm}(x,0;v)\in P(H(v))$. Moreover, if $u(x)$ has two derivatives $u',u''\in L^1(\boldsymbol{R})$, then $v(x)$ also has two derivatives $v',v''\in L^1(\boldsymbol{R})$. Hence, the formulas (1), (5) and (6) are valid for the potential $v(x)$. By Theorem 1, we have

$$f_{+}(x,\xi;u)=i\xi^{-1}(-f'_{+}(x,\xi;v)+q(x)f_{+}(x,\xi;v)),$$

and $r_{+}(\xi;u)=-r_{+}(\xi;v)$, where $q(x)=(d/dx)\log f_{+}(x,0;v)$. Hence, by direct calculation, one verifies

$$2i\pi^{-1}\int_{-\infty}^{\infty}\xi r_{+}(\xi;u)f_{+}(x,\xi;u)^2d\xi=\sum_{j=1}^{3}F_j(x;v),$$

where $F_j(x;v)$, $1\leqq j\leqq 3$, are defined as follows:

$$F_1(x;v)=2i\pi^{-1}\int_{-\infty}^{\infty}\xi^{-1}r_{+}(\xi;v)f'_{+}(x,\xi;v)^2d\xi,$$

$$F_2(x;v)=-4i\pi^{-1}\int_{-\infty}^{\infty}\xi^{-1}r_{+}(\xi;v)f_{+}(x,\xi;v)f'_{+}(x,\xi;v)d\xi,$$

$$F_3(x;v)=2i\pi^{-1}q(x)^2\int_{-\infty}^{\infty}\xi^{-1}r_{+}(\xi;v)f_{+}(x,\xi;v)^2d\xi.$$

By (1), (5) and (6) for $v(x)$, we have immediately

$$F_1(x;v)=-v(x),\quad F_2(x;v)\equiv 0,\quad F_3(x;v)=2q(x)^2.$$

On the other hand, one can show easily

$$u(x)=v^*(x;f_{+})=-v(x)+2q(x)^2.$$

This implies that the formula (1+) holds for $u(x)\in\Gamma^{(2)}_{+}$ with $u',u''\in L^1(\boldsymbol{R})$. The proof of (1−) for $u(x)\in\Gamma^{(2)}_{+}$ is similar. Moreover, by the parallel method, one can show the formula (1) for $u(x)\in\Gamma_-^{(2)}$ with $u', u''\in L^1(\mathbf{R})$. Thus we have the following.

Theorem 2. *The Deift-Trubowitz trace formula* (1) *is valid also for the potential* $u(x)\in\Gamma_\pm^{(2)}$ *with* $u', u''\in L^1(\mathbf{R})$.

4. Miscellaneous formulas. Next we will derive the formulas corresponding to (5) and (6) in our case. Define $\phi_j(x;\pm)$ by

$$\phi_1(x;\pm)=i\pi^{-1}\int_{-\infty}^{\infty}\xi^{-1}r_\pm(\xi;u)f_\pm(x,\xi;u)^2d\xi, \tag{7}$$

$$\phi_2(x;\pm)=i\pi^{-1}\int_{-\infty}^{\infty}\xi^{-1}r_\pm(\xi;u)f'_\pm(x,\xi;u)^2d\xi. \tag{8}$$

If we assume $u(x)\in\Gamma_\pm^{(3)}$ with $u', u''\in L^1(\mathbf{R})$, then the integrals in (7) and (8) converge as principal values, respectively. One verifies that ϕ_1 solves the 3-rd order differential equation

$$\phi_1'''-4u\phi_1'-2u'\phi_1=-2u'. \tag{9}$$

Moreover, since $f_\pm(x,\xi;u)$ behave like $\exp(\pm i\xi x)$ as $x\to\pm\infty$ respectively, and $\overline{r_\pm(\xi;u)}=r_\pm(-\xi;u)$, one can show that $\phi_1(x;\pm)$ tend to -1 as $x\to\pm\infty$ respectively by the Riemann-Lebesgue theorem. On the other hand one has

Lemma 3. *Let $f(x)$ and $g(x)$ be solutions of*

$$-y''+u(x)y=0, \tag{10}$$

then the product $f(x)g(x)$ solves

$$y'''-4u(x)y'-2u'y=0, \tag{11}$$

which is the homogeneous equation associated with (9). *Moreover*

$$W(f^2, fg, g^2)=2W(f,g)^2 \tag{12}$$

holds, where $W(f_1,\cdots,f_n)=\det(\partial^{i-1}f_j)_{1\leq i,j\leq n}$ are the Wronskians.

Moreover, we have

Lemma 4. *If $u(x)\in\Gamma_\pm^{(2)}$ then $f_\pm(x,0;u)$ exist, and $\lim_{x\to\pm\infty}f_\pm(x,0;u)=1$ hold, i.e., $f_\pm(x,0;u)\in S_\pm(u)$, respectively.*

On the other hand, it is shown in [4; Theorem 2, p. 18] that if $u(x)\in\Pi_0$ and $H(u)$ has no bound states then $S_\pm(u)\subset P(H(u))$ follows. Hence, by Lemma 4, if $u(x)\in\Gamma_\pm^{(3)}$ then $f_\pm(x,0;u)\in P(H(u))$ follows. Now suppose $u(x)\in\Gamma_+^{(3)}$. Put $f_1(x)=f_+(x,0;u)$ and

$$f_2(x)=f_+(x,0;u)\int_0^x f_+(x,0;u)^{-2}dx.$$

Then $f_1(x)$ and $f_2(x)$ are the fundamental system of solutions of (10) such that $W(f_1,f_2)=1$. Hence, by Lemma 3, $g_1(x)=f_1(x)^2$, $g_2(x)=f_1(x)f_2(x)$ and $g_3(x)=f_2(x)^2$ are the fundamental system of solutions of (11). One verifies that $g_1(x)$ tends to 1 as $x\to\infty$, and

$$g_2(x)=O(x),\quad g_3(x)=O(x^2)\quad \text{as } x\to\infty.$$

On the other hand, because the constant 1 is a particular solution of (9), by taking into consideration the asymptotic behaviours of ϕ_1 and g_j, we have

$$\phi_1(x;\pm)=1-2g_1(x)=1-2f_+(x,0;u)^2.$$

A similar consideration is valid also for $u(x)\in\Gamma_-^{(3)}$ with $u', u''\in L^1(\mathbf{R})$. Thus we have

Theorem 5. *If $u(x) \in \Gamma_{\pm}^{(3)}$ with $u', u'' \in L^1(\boldsymbol{R})$ then the formulas*

$$(13\pm) \qquad i\pi^{-1}\int_{-\infty}^{\infty} \xi^{-1} r_{\pm}(\xi\,;u) f_{\pm}(x,\xi\,;u)^2 d\xi + 2 f_{\pm}(x,0\,;u)^2 = 1,$$

$$(14\pm) \qquad -2i\pi^{-1}\int_{-\infty}^{\infty} \xi^{-1} r_{\pm}(\xi\,;u) f'_{\pm}(x,\xi\,;u)^2 d\xi - 4 f'_{\pm}(x,0\,;u)^2 = u(x)$$

are valid, respectively.

If $u(x) \in \Gamma_{\pm}^{(2)}$ then $f_{\mp}(x,\xi\,;u) = O(1/\xi)$ as $\xi \to 0$. Hence (13±) and (14±) have no meaning for $u(x) \in \Gamma_{\mp}^{(3)}$ respectively.

The detailed proof will appear elsewhere.

References

[1] P. Deift, F. Lund, and E. Trubowitz: Nonlinear wave equations and constrained harmonic motion. Comm. Math. Phys., **74**, 141–188 (1980).

[2] P. Deift and E. Trubowitz: Inverse scattering on the line. Comm. Pure Appl. Math., **32**, 121–251 (1979).

[3] M. Ohmiya: On the Darboux transformation of the 1-dimensional Schrödinger operator and Levinson's theorem. J. Math. Tokushima Univ., **21**, 13–26 (1987).

[4] ——: On the inverse scattering problem for 1-dimensional Schrödinger operator with integrable potential. ibid., **22**, 15–28 (1988).