56. *Inverse Map Theorem in the Ultra-F-differentiable Class*

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Let M_p, $p=0, 1, 2, \cdots$, be a sequence of positive numbers. We assume that $M_0=1$ and that there is a constant $H\geqq 1$ such that

(1) $$(M_p/p\,!)^{1/p}\leqq H(M_q/q\,!)^{1/q}, \qquad \text{if } 1\leqq p\leqq q.$$

Let X, Y be Banach spaces and $U\subset X$ be an open set. A C^∞-map (in the sense of Frechét derivatives) $f: U\to Y$ is said to belong to the ultra-F-differentiable class $\alpha\{M_p\}$, if there are constants $C\geqq 0$ and $h>0$ such that, if $x\in U$ and $p\geqq 0$, then $\|f^{(p)}(x)\|\leqq Ch^pM_p$.

The main purpose of this note is to give an inverse map theorem (Theorem 2) in the class $\alpha\{M_p\}$ which is an improvement and a generalization of a similar theorem by H. Komatsu [2] in the following sense. Our result improves that of [2] in the sense that the condition (1) is simpler and a little weaker than the corresponding one in [2]

(2) $$(M_p/p\,!)^{1/(p-1)}\leqq H(M_q/q\,!)^{1/(q-1)}, \qquad \text{if } 2\leqq p\leqq q.$$

Also, our theorem generalizes to the infinite dimensional case that of [2] which deals with the finite dimensional case. In order to prove his inverse map theorem Komatsu used majorant series. Essentially the same idea is used in this note, too. But in order to deal with the infinite dimensional case in a clear-cut way we resort to a convenient tool that was not used in [2]. It is the higher order chain rule of Faa' di Bruno [1], which reads, in the one dimensional case,

(3) $$(f\circ g)^{(p)}(x)=p\,!\sum_{j=1}^{p}f^{(j)}(g(x))\sum_{|q|=j\ \|q\|=p}\prod_{i=1}^{p-j+1}\frac{1}{q_i\,!}\left(\frac{g^{(i)}(x)}{i\,!}\right)^{q_i},$$

where f and g are real-valued functions of a real variable, $q=(q_1, \cdots, q_{p-j+1})$ is a multi-index and

$$|q|=q_1+q_2+\cdots+q_{p-j+1}, \qquad \|q\|=q_1+2q_2+\cdots+(p-j+1)q_{p-j+1}.$$

Let us first generalize the above rule to the infinite dimensional case. In the following theorem the symbol sym denotes the symmetrization of a multi-linear operator. For its definition and other notational conventions with respect to multi-linear operators see [3].

Theorem 1. *Let X, Y, Z be normed spaces and $U\subset X$, $V\subset Y$ be open. Let $g: U\to V$ and $f: V\to Z$ be C^p-maps. Then for $x\in U$*

(4) $$(f\circ g)^{(p)}(x)=\mathrm{sym}\left(p\,!\sum_{j=1}^{p}f^{(j)}(g(x))\sum_{|q|=j\ \|q\|=p}\prod_{i=1}^{p-j+1}\frac{1}{q_i\,!}\left\{\frac{g^{(i)}(x)}{i\,!}\right\}^{q_i}\right).$$

Proof. Let x be an arbitrary fixed point of U. Put $y=g(x)$. For $h\in X$ with small norm we have, by Taylor's rule,

$$
\begin{aligned}
(5)\quad f(g(x+h))-f(g(x)) &= f(g(x+h))-f(y)\\
&=\sum_{j=1}^{p}\frac{f^{(j)}(y)}{j!}\{g(x+h)-g(x)\}^{j}+o(\|g(x+h)-g(x)\|^{p})\\
&=\sum_{j=1}^{p}\frac{f^{(j)}(y)}{j!}\left\{\sum_{i=1}^{p-j+1}\frac{g^{(i)}(x)}{i!}h^{i}+o(\|h\|^{p-j+1})\right\}^{j}+o(\|h\|^{p})\\
&=\sum_{j=1}^{p}\frac{f^{(j)}(y)}{j!}\left\{\sum_{i=1}^{p-j+1}\frac{g^{(i)}(x)}{i!}h^{i}\right\}^{j}+o(\|h\|^{p})\\
&=\sum_{j=1}^{p}\frac{f^{(j)}(y)}{j!}j!\sum_{|q|=j}\prod_{i=1}^{p-j+1}\frac{1}{q_i!}\left\{\frac{g^{(i)}(x)}{i!}h^{i}\right\}^{q_i}+o(\|h\|^{p})\\
&=\sum_{j=1}^{p}f^{(j)}(y)\sum_{|q|=j}\prod_{i=1}^{p-j+1}\frac{1}{q_i!}\left\{\frac{g^{(i)}(x)}{i!}\right\}^{q_i}h^{\|q\|}+o(\|h\|^{p})\\
&=\sum_{j=1}^{p}f^{(j)}(y)\sum_{k=1}^{p}\left(\sum_{|q|=j\ \|q\|=k}\prod_{i=1}^{p-j+1}\frac{1}{q_i!}\left\{\frac{g^{(i)}(x)}{i!}\right\}^{q_i}\right)h^{k}+o(\|h\|^{p})\\
&=\sum_{k=1}^{p}\left(\sum_{j=1}^{p}f^{(j)}(y)\sum_{|q|=j\ \|q\|=k}\prod_{i=1}^{p-j+1}\frac{1}{q_i!}\left\{\frac{g^{(i)}(x)}{i!}\right\}^{q_i}\right)h^{k}+o(\|h\|^{p})\\
&=\sum_{k=1}^{p}\mathrm{sym}\left(\sum_{j=1}^{p}f^{(j)}(y)\sum_{|q|=j\ \|q\|=k}\prod_{i=1}^{p-j+1}\frac{1}{q_i!}\left\{\frac{g^{(i)}(x)}{i!}\right\}^{q_i}\right)h^{k}+o(\|h\|^{p}).
\end{aligned}
$$

On the other hand we have

$$
(6)\qquad f(g(x+h))-f(g(x))=\sum_{k=1}^{p}\frac{(f\circ g)^{(k)}(x)}{k!}h^{k}+o(\|h\|^{p}).
$$

Comparing (5) with (6) we see that the equality

$$
\frac{(f\circ g)^{(k)}(x)}{k!}=\mathrm{sym}\left(\sum_{j=1}^{p}f^{(j)}(y)\sum_{|q|=j\ \|q\|=k}\prod_{i=1}^{p-j+1}\frac{1}{q_i!}\left\{\frac{g^{(i)}(x)}{i!}\right\}^{q_i}\right).
$$

holds for $k=1,\cdots,p$. In particular we have (4). Q.E.D.

By means of the above result we can now prove the following

Theorem 2. *Let X and Y be Banach spaces, $U\subset X$ and $V\subset Y$ be open sets and $f: U\to V$ be a C^{∞}-map. Assume that the derivative $f': U\to L(X,Y)$ of f belongs to the ultra-F-differentiable class $\alpha\{M_p\}$. Let x_0 be a point of U and assume that the inverse $[f'(x_0)]^{-1}\in L(Y,X)$ exists. Then there exist open sets U_0 and V_0 such that $x_0\in U_0\subset U$, $f(x_0)\in V_0\subset V$ and f is a C^{∞}-diffeomorphism from U_0 onto V_0 and the derivative $(f^{-1})'$ of the inverse map $f^{-1}: V_0\to U_0$ of f belongs to the class $\alpha\{M_p\}$.*

Proof. Since f is a C^{∞}-map on U and $[f'(x_0)]^{-1}$ exists, we know by the well-known inverse map theorem for general C^{∞}-maps that there exist open sets U_0 and V_0 such that $x_0\in U_0\subset U$, $f(x_0)\in V_0\subset V$ and f is a C^{∞}-diffeomorphism from U_0 onto V_0. We denote by g the inverse map of $f: U_0\to V_0$.

Take a point $a\in U_0$ and fix it. Set $b=f(a)$ and write $S=f'(a)$, $T=S^{-1}=g'(b)$. Next put

$$
(7)\qquad \phi(x)=x-Tf(x),\qquad x\in U_0.
$$

Then we have for $y\in V_0$

$$
(8)\qquad g(y)=Ty+(\phi\circ g)(y).
$$

Since $\phi'(a)=0$, (8) implies $g'(b)=T$. If $p\geqq 2$, we have, by the Faa' di

Bruno rule (4),

$$g^{(p)}(b)=\mathrm{sym}\left(p!\sum_{j=2}^{p}\phi^{(j)}(a)\sum_{|q|=j\ \|q\|=p}\prod_{i=1}^{p-j+1}\frac{1}{q_i!}\left\{\frac{g^{(i)}(x)}{i!}\right\}^{q_i}\right)$$

and accordingly

$$\|g^{(p)}(b)\|\leqq p!\sum_{j=2}^{p}\|\phi^{(j)}(a)\|\sum_{|q|=j\ \|q\|=p}\prod_{i=1}^{p-j+1}\frac{1}{q_i!}\left\{\frac{\|g^{(i)}(x)\|}{i!}\right\}^{q_i}. \tag{9}$$

Since f' belongs to $\alpha\{M_p\}$, there are constants $C\geqq 0$ and $h>0$ such that, if $x\in U_0$ and $j\geqq 1$,

$$\|f^{(j)}(x)\|\leqq Ch^{j-1}M_{j-1}. \tag{10}$$

Hence, if $j\geqq 2$, we have

$$\|\phi^{(j)}(x)\|\leqq C\|T\|h^{j-1}M_{j-1}. \tag{11}$$

Now take a constant $A\geqq\|T\|$ arbitrarily and let N be an integer $\geqq 2$. We put for $t\in\mathbf{R}$

$$\psi_N(t)=CA\sum_{j=2}^{N}\frac{h^{j-1}M_{j-1}}{j!}t^j. \tag{12}$$

We want to solve the equation $s=(t-\psi_N(t))/A$ with respect to t, i.e., we seek a function

$$g_N(s)=c_1s+c_2s^2+\cdots \tag{13}$$

such that the identity

$$s=(g_N(s)-\psi_N(g_N(s)))/A \tag{14}$$

holds for all s with small $|s|$. By the Lagrange expansion theorem (Goursat, Cours d'analyse, t.1, chap, IX, n ∘ 195) the coefficients $c_1, c_2, \cdots$ in (13) are given by the formula

$$c_i=\frac{1}{i!}\left(\frac{d^{i-1}}{dt^{i-1}}\left\{\frac{A}{1-\psi_N(t)/t}\right\}\right)_{t=0}. \tag{15}$$

On the other hand (14) is rewritten as

$$g_N(s)=As+(\psi_N\circ g_N)(s). \tag{16}$$

Hence, if $p\geqq 2$, we have, by the Faa' di Bruno rule (3),

$$\begin{aligned} g_N^{(p)}(0)&=p!\sum_{j=2}^{p}\phi_N^{(j)}(0)\sum_{|q|=j\ \|q\|=p}\prod_{i=1}^{p-j+1}\frac{1}{q_i!}\left\{\frac{g_N^{(i)}(0)}{i!}\right\}^{q_i}\\ &=p!\sum_{j=2}^{p}CAM_{j-1}h^j\sum_{|q|=j\ \|q\|=p}\prod_{i=1}^{p-j+1}\frac{1}{q_i!}c_i^{q_i}. \end{aligned} \tag{17}$$

Using (15) we can show that c_i with $2\leqq i\leqq N$ satisfies

$$0<c_i<A\{4A(CA+1)Hh\}^{i-1}M_{i-1}/i!. \tag{18}$$

In fact, if $2\leqq i\leqq N$, we have, by (1),

$$\begin{aligned} 0<c_i&=\frac{A^i}{i!}\left(\frac{d^{i-1}}{dt^{i-1}}\left\{\sum_{s=0}^{\infty}\left(CA\sum_{r=1}^{i-1}\frac{M_r}{(r+1)!}(ht)^r\right)^s\right\}^i\right)_{t=0}\\ &<\frac{A^i}{i!}\left(\frac{d^{i-1}}{dt^{i-1}}\left\{\sum_{s=0}^{\infty}\left(CA\sum_{r=1}^{i-1}\frac{M_r}{r!}(ht)^r\right)^s\right\}^i\right)_{t=0}\\ &<\frac{A^i}{i!}\left(\frac{d^{i-1}}{dt^{i-1}}\left\{\sum_{s=0}^{\infty}\left(CA\sum_{r=1}^{\infty}\left\{H\left(\frac{M_{i-1}}{(i-1)!}\right)^{1/(i-1)}ht\right\}^r\right)^s\right\}^i\right)_0. \end{aligned}$$

Further we continue similar calculation to that in [2], p. 70, and obtain (18).

Next we compare $\|g^{(p)}(b)\|$ with $g_N^{(p)}(0)$ for $1 \leqq p \leqq N$. First note, by (8) and (16), that

(19) $$\|g'(b)\| = \|T\| \leqq A = g'_N(0).$$

Also we note, by (11) and (12), that

(20) $$\|\phi^{(j)}(a)\| \leqq \psi_N^{(j)}(0), \qquad 2 \leqq j \leqq N.$$

Using (9), (17), (19) and (20) it is easy to show inductively that the inequalities

$$\|g^{(p)}(b)\| \leqq g_N^{(p)}(0) = p!\, c_p < A\{4A(CA+1)Hh\}^{p-1} M_{p-1}$$

hold for $2 \leqq p \leqq N$. But N is an arbitrary integer $\geqq 2$. Hence we see that the inequality

(21) $$\|g^{(p)}(b)\| < A\{4A(CA+1)Hh\}^{p-1} M_{p-1}$$

holds for all $p \geqq 2$, which, together with (19) and the condition $M_0 = 1$, shows that g' belongs to the class $\alpha\{M_p\}$. Q.E.D.

From the above inverse map theorem we can deduce the following implicit map theorem.

Theorem 3. *Let X, Y and Z be Banach spaces, $U \subset X$ and $V \subset Y$ be open sets and $f: U \times V \to Z$ be a C^∞-map such that $f(a, b) = 0$ for some point $(a, b) \in U \times V$ and $[\partial_2 f(a, b)]^{-1} \in L(Z, Y)$ exists. Assume that there exist constants $C \geqq 0$ and $h > 0$ such that*

$$\|\partial_1^p \partial_2^q f(x, y)\| \leqq C h^{p+q-1} M_{p+q-1}$$

for $p+q \geqq 1$ and $(x, y) \in U \times V$. Then there exist open sets U_0 and V_0 such that $a \in U_0 \subset U$, $b \in V_0 \subset V$ and that there exists a unique C^∞-map $g: U_0 \to V_0$ such that $g(a) = b$, $f(x, g(x)) = 0$ for all $x \in U_0$ and g' belongs to the class $\alpha\{M_p\}$.

A proof of the above theorem is given by the standard technique to consider the inverse map of the map $(x, y) \mapsto (x, f(x, y))$. We omit the detail.

References

[1] Faa' di Bruno, F.: Sullo sviluppo delle funzioni. Annali di Scienze Matematiche e Fisiche di Tortolini, **6**, 479–489 (1855).

[2] Komatsu, H.: The implicit function theorem for ultradifferentiable mappings. Proc. Japan Acad., **55A**, 69–72 (1979).

[3] Yamanaka, T.: A New higher order chain rule and Gevrey class. Annales of Global Analysis and Geometry, vol. 7, pp. 181–205 (1989).