44. On the Inverse Scattering on the Line and the Darboux Transformation

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In this paper we study the inverse scattering problem for the 1-dimensional Schrödinger operator

$$H(u)=-\frac{d^2}{dx^2}+u(x), \qquad -\infty<x<\infty$$

by the method of the Darboux transformation. Here we assume that the potential $u(x)$ belongs to

$$L_{1,\lambda}=\left\{u \,|\, \text{real valued, continuous and } \int_{-\infty}^{\infty}|x|^{\lambda}|u(x)|\,dx<\infty\right\}$$

for some $\lambda\geqq 0$. In this article, we omitted the proof. See [3] and [4] for details.

1. Jost solutions. Let $f_\pm(x,\xi\,;u)$ be the solutions of the eigenvalue problem

$$H(u)f_\pm=-f''_\pm+u(x)f_\pm=\xi^2 f_\pm$$

such that $f_\pm(x,\xi\,;u)$ behave like $e^{\pm i\xi x}$ as $x\to\pm\infty$ respectively, which are called the Jost solutions, if they exist. If $u(x)\in L_{1,0}$, then $f_\pm(x,\xi\,;u)$ exist for $\xi\in \boldsymbol{R}\backslash\{0\}$. Moreover, if $u(x)\in L_{1,1}$, then $f_\pm(x,\xi\,;u)$ extended analytically into the complex upper half plane $\operatorname{Im}\xi>0$. More precisely, $e^{\mp i\xi x}f_\pm(x,\xi\,;u)-1$ belong to the Hardy space H^{2+} of the upper half plane and, therefore, they admit the integral representation

$$e^{\mp i\xi x}f_\pm(x,\xi\,;u)=1\pm\int_0^{\pm\infty}B_\pm(x,y)e^{\pm i\xi y}dy. \tag{1}$$

In particular, $f_\pm(x,0\,;u)$ are defined. The entries of the S-matrix of $H(u)$ are represented explicitly in terms of the Jost solutions. For example, we have

$$r_\pm(\xi\,;u)=\frac{[f_+(x,\mp\xi\,;u),f_-(x,\pm\xi\,;u)]}{[f_-(x,\xi\,;u),f_+(x,\xi\,;u)]},$$

where $r_+(\xi\,;u)$ and $r_-(\xi\,;u)$ are the right and left reflection coefficients respectively, and $[f,g]=fg'-gf'$ is the Wronskian. We refer to [1] for explicit representations of another entries and further information about the scattering data.

2. Levinson's theorem. The following, which is called Levinson's theorem usually, is well known.

Theorem 1 (cf. [1; p. 208]). *A potential $u(x)$ in $L_{1,1}$ without bound states is determined by its right reflection coefficient.*

On the other hand, it is shown in [3] that such uniqueness is not valid for the potential $u(x)$ in $L_{1,0}$. More precisely, we have the following.

Theorem 2 (cf. [3; p. 25]). *There exist $u(x)$ and $v(x)$ in $L_{1,0}\backslash L_{1,1}$ such that $u(x)\neq v(x)$, $H(u)$ and $H(v)$ have no bound states, and their right reflection coefficients coincide with each other.*

We can prove Theorem 2 by constructing such potentials by the method of the Darboux transformation. Here we explain the Darboux transformation. Let $P(u)$ be the set of all positive solutions of the differential equation

$$H(u)f=-f''+u(x)f=0 \tag{2}$$

and suppose $f(x)\in P(u)\neq\emptyset$. Put $A_f=d/dx+f'/f$ then $H(u)=A_fA_f^*$ follows, where A_f^* is the formal adjoint of A_f. We define the Darboux transformation $H^*(u;f)$ by $H^*(u;f)=A_f^*A_f$. Put

$$u^*=u^*(x;f)=u(x)-2(\log f(x))'',$$

then $H^*(u;f)=H(u^*)$ follows.

3. Positive solutions. In this section we discuss whether the equation (2) has positive solutions or not. Define $S_\pm(u)$ by

$$S_\pm(u)=\{f \mid \text{solutions of (2), and } \exists \lim_{x\to\pm\infty} f(x)\in(0,\infty)\}$$

respectively. In [1], Deift and Trubowitz showed that if $u(x)$ is in $L_{1,2}$, and $H(u)$ has no bound states, then $f_\pm(x,0;u)$ belong to $P(u)$. On the other hand, we have

Theorem 3 (cf. [4; Theorem 2]). *If $u(x)$ is in $L_{1,0}$, and $H(u)$ has no bound states, then $S_\pm(u)\subset P(u)$ follows.*

Theorem of Deift-Trubowitz mentioned above can be obtained as a corollary of Theorem 3. Put $S(u)=S_+(u)\cup S_-(u)$, then we have

Theorem 4 (cf. [4]). *Suppose that $u(x)\in L_{1,0}$, $H(u)$ has no bound states and $S(u)\neq\emptyset$. Put $u^*=u^*(x;f)$ for $f(x)\in S(u)$. Then the Jost solutions $f_\pm(x,\xi;u^*)$ exist for all $\xi\in \boldsymbol{R}\backslash\{0\}$. Moreover,*

$$r_\pm(\xi;u^*)=-r_\pm(\xi;u)$$

are valid.

Here we prove Theorem 2. Suppose that $w(x)$ is in $L_{1,2}$, and $r_\pm(0;w)=-1$ (this holds if and only if $f_\pm(x,0;w)$ are linearly independent). Moreover assume that $H(w)$ has no bound states. Put

$$u(x)=w(x)-2(\log f_+(x,0;w))'' \tag{3}$$

and

$$v(x)=w(x)-2(\log f_-(x,0;w))''. \tag{4}$$

Then, it follows that $u(x)\neq v(x)$, $u(x)$ and $v(x)$ are in $L_{1,0}\backslash L_{1,1}$, $r_\pm(\xi;u)=r_\pm(\xi;v)=-r_\pm(\xi;w)$, and $H(u)$ and $H(v)$ have no bound states.

4. Inverse problem. Suppose that the function $r(\xi)$ ($\xi\in\boldsymbol{R}$) is continuous, $|r(\xi)|<1$ for all $\xi\in\boldsymbol{R}\backslash\{0\}$, $r(\xi)=O(1/\xi)$ as ξ tends to $\pm\infty$, $r(0)=1$, $\overline{r(\xi)}=r(-\xi)$, the Fourier transform $\tilde{r}(x)$ of $r(\xi)$ is absolutely continuous, and

$$\int_\alpha^\infty (1+x^2)\left|\frac{d}{dx}\tilde{r}(x)\right|dx<\infty \qquad \text{for all } \alpha.$$

Then, by the inverse scattering theory for potentials in $L_{1,2}$ (cf. [2]) it turns out there exists uniquely the potential $w(x)$ in $L_{1,2}$ such that $r_+(\xi; w) = -r(\xi)$, and $H(w)$ has no bound states. Next, define $u(x)$ and $v(x)$ by (3) and (4). Then, from Theorems 2 and 4, it follows that $u(x) \neq v(x)$, $u(x)$ and $v(x)$ belong to $L_{1,0} \backslash L_{1,1}$, $r_+(\xi; u) = r_+(\xi; v) = r(\xi)$, and $H(u)$ and $H(v)$ have no bound states. Moreover, it follows from Darboux's lemma (cf. [5; p. 88] and [4; Lemma 1]) that $1/f_+(x, 0; w)$ and $1/f_-(x, 0; w)$ belong to $S_+(u)$ and $S_-(v)$ respectively. By combining Theorems 1 and 4, we can show that $u(x)$ and $v(x)$ are the only potentials in $L_{1,0}$ such that their right reflection coefficient coincide with $r(\xi)$, $H(u)$ and $H(v)$ have no bound states, and $S_+(u)$ and $S_-(v)$ are non-empty.

5. Concluding remark. The inverse problem of the scattering theory without bound states is usually devided into the following three parts (cf. [1; p. 122]):

I. Uniqueness; Does the reflection coefficient determine the potential?

II. Reconstruction; Give an algorithm for recovering the potential from the reflection coefficient.

III. Characterization; Give necessary and sufficient conditions for a given 2×2 matrix to be the S-matrix of a potential.

By Levinson's theorem, the answer to problem I is yes, if the potential is in $L_{1,1}$. Problems II and III for $L_{1,2}$ were solved by Faddeev [2] and Deift-Trubowitz [1] respectively.

On the other hand, none of these problems for $L_{1,0}$ have been explored. The purpose of the present work is to solve these problems by restricting our attention to the potential $u(x)$ in $L_{1,0} \backslash L_{1,1}$ such that $H(u)$ has no bound states, and $S_+(u)$ (or $S_-(u)$) is not void.

References

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