5. *On the Total Variation of Argument $f(z)$ Whose Derivative Has a Positive Real Part*

By Mamoru NUNOKAWA
Department of Mathematics, Gunma University

(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1989)

1. Introduction. Let R denote the class of functions which are analytic and satisfy $Ref'(z)>0$ for $|z|<1$ and are normalized by $f(0)=0$ and $f'(0)=1$.

Noshiro [3] and Warschawski [4] showed that $Ref'(z)>0$ is a sufficient condition for the univalence of $f(z)$ in any convex domain.

MacGregor [2] investigated the class of functions which belong to R and obtained many interesting results.

It is the purpose of the present paper to obtain the total variation of argument $f(z)$ whose derivative has a positive real part.

2. Preliminaries. Lemma 1. *Let $f(z)\in R$, then*

$$\left|\frac{z}{f(z)}\right|\leqq\frac{2}{r}\log(1+r)-1$$

where $0<|z|=r<1$.

We owe this lemma to [2, Theorem 1].

Lemma 2. *Let $f(z)\in R$, then*

$$\int_0^{2\pi}|f'(z)|d\theta\leqq 2\pi+4\log\frac{1+r}{1-r}$$

where $|z|=r<1$.

We owe this lemma to [1, p. 482].

3. Statement of result. Theorem. *Let $f(z)\in R$, then*

$$(1)\qquad \int_0^{2\pi}\left|\mathrm{Re}\frac{zf'(z)}{f(z)}\right|d\theta\leqq 2\pi+4\log\frac{1+r}{1-r}$$

where $|z|=r<1$.

Proof. From Lemmas 1 and 3, we easily have

$$\begin{aligned}\int_0^{2\pi}\left|\mathrm{Re}\frac{zf'(z)}{f(z)}\right|d\theta&\leqq\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|d\theta\leqq\left(\frac{2}{r}\log(1+r)-1\right)\int_0^{2\pi}|f'(z)|d\theta\\&\leqq(2\log(1+r)^{1/r}-1)\left(2\pi+4\log\frac{1+r}{1-r}\right)\\&\leqq 2\pi+4\log\frac{1+r}{1-r}\end{aligned}$$

where $0<|z|=r<1$.

For the case $r=0$, the estimation (1) is true.

This completes our proof and (1) implies that

$$\int_0^{2\pi}\left|\mathrm{Re}\frac{zf'(z)}{f(z)}\right|d\theta=0\left(\log\frac{1}{1-r}\right)\qquad as\ r\to 1.$$

This means the order of infinity of the total variation of argument $f(z)\in R$ is at most $\log 1/(1-r)$ as $r\to 1$.

The author can not answer the question whether there is a function $f(z)\in R$ for which

$$\lim_{r\to 1}\frac{V(r)}{\log 1/(1-r)}>0$$

where

$$V(r)=\int_0^{2\pi}\left|\mathrm{Re}\frac{zf'(z)}{f(z)}\right|d\theta,\quad |z|=r<1.$$

References

[1] F. R. Keogh: Some theorems on conformal mapping of bounded star-shaped domains. Proc. London Math. Soc., (3) **9**, 481–491 (1959).
[2] T. H. MacGregor: Functions whose derivative has a positive real part. Trans. Amer. Math. Soc., **104**, 532–537 (1962).
[3] K. Noshiro: On the theory of schlicht functions. J. Fac. Sci. Hokkaido Univ., (1) **2**, 129–155 (1934–1935).
[4] S. Warschawski: On the higher derivatives at the boundary in conformal mappings. Trans. Amer. Math. Soc., **38**, 310–340 (1935).