38. A Discrepancy Problem with Applications to Linear Recurrences. I

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1. **Introduction.** Let $R=\{R_n\}_{n=0}^{\infty}$ be a second order linear recursive sequence of rational integers defined by

$$R_n=A\cdot R_{n-1}+B\cdot R_{n-2}\ (n>1),$$

where the initial values R_0, R_1 and A, B are fixed integers. We suppose that $AB\neq 0$, $R_0^2+R_1^2\neq 0$ and $D=A^2+4B\neq 0$. It is well-known that the terms of R can be expressed as

$$R_n=a\cdot\alpha^n-b\cdot\beta^n \tag{1}$$

for any $n\geqq 0$, where α and β are the roots of the polynomial x^2-Ax-B and

$$a=\frac{R_1-R_0\beta}{\alpha-\beta},\qquad b=\frac{R_1-R_0\alpha}{\alpha-\beta}.$$

Throughout this paper, we assume $|\alpha|\geqq|\beta|$ and that the sequence is non-degenerate, i.e. α/β is not a root of unity. We may also suppose that $R_n\neq 0$ for $n>0$ since in [2] it was proved that a non-degenerate sequence R has at most one zero term and after a movement of indices this condition will be fulfilled.

If $D=A^2+4B>0$, i.e. if α and β are real numbers, then $|\alpha|>|\beta|$ and $(\beta/\alpha)^n\to 0$ as $n\to\infty$; hence we obtain by (1)

$$\lim_{n\to\infty}(R_{n+1}/R_n)=\alpha. \tag{2}$$

The following interesting problem arises: what is the quality of approximation of α by rationals of the form R_{n+1}/R_n? In the case $D>0$ we know that there are constants $q>0$ and k_0 $(0<k_0\leqq 2)$, depending on the parameters of the sequence R, such that

$$\left|\alpha-\frac{R_{n+1}}{R_n}\right|<q\cdot R_n^{-k} \tag{3}$$

for infinitely many n and for any $k\leqq k_0$, but (3) holds only for finitely many n if $k>k_0$ (see [5]). For the sequence R with initial values $R_0=0$, $R_1=1$ it was proved in [3] that $k_0=2$ if and only if $|B|=1$; furthermore

$$\left|\alpha-\frac{R_{n+1}}{R_n}\right|<\frac{1}{\sqrt{D}\cdot R_n^2}$$

for infinitely many n, and these rational numbers R_{n+1}/R_n give the best

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possible approximation of α.

If $D<0$, i.e. if α and β are non-real complex numbers with $|\alpha|=|\beta|$, then (2) does not hold, even if we consider the absolute values of the numbers. For this case we introduce some notations. Since $\beta=\bar{\alpha}$ and $-b=\bar{a}$ are complex conjugates of α and a respectively, we can write

$$\alpha=r\cdot e^{\pi\theta i},\quad \beta=r\cdot e^{-\pi\theta i},\quad \frac{\alpha}{\beta}=e^{2\pi\theta i}$$

and

$$a=r_1\cdot e^{2\pi\gamma i},\quad -b=r_1\cdot e^{-2\pi\gamma i},\quad \frac{a}{b}=e^{2\pi\omega i},$$

where θ, γ and ω are real numbers with $0<\theta, \gamma, \omega<1$. By (1), using the fact $|\alpha|=|\beta|$, we have

$$(4)\qquad \left|\frac{R_{n+1}}{R_n}\right|=|\beta|\cdot\left|\frac{(a/b)(\alpha/\beta)^{n+1}-1}{(a/b)(\alpha/\beta)^n-1}\right|=|\alpha|\cdot\left|\frac{e^{2\pi(n+1)\theta i+2\pi\omega i}-1}{e^{2\pi n\theta i+2\pi\omega i}-1}\right|.$$

By our conditions α/β is not a root of unity and so θ is an irrational number. This implies that the sequence $(n\theta+\omega)$, $n=1,2,3,\cdots$ is uniformly distributed modulo 1 and by (4) it is easy to see that

$$(5)\qquad \left||\alpha|-\left|\frac{R_{n+1}}{R_n}\right|\right|<\varepsilon$$

for any $\varepsilon>0$ and for infinitely many n.

In this paper we undertake further investigations of the approximation of $|\alpha|$ by rational numbers of the form $|R_{n+1}/R_n|$ as in (5). First, we show a result for the discrepancy of the sequence $(n\theta+\omega)$ and then we apply it to show that ε can be chosen as $\varepsilon=n^{-c}=O((\log|R_n|)^{-c})$, where c is a constant depending on the sequence R. We shall also show (in Theorem 3) that, apart from the constant c, it is the best possible.

Finally we note that in [4] we also studied the sequence R_{n+1}/R_n with $D<0$ and real parameters; we have shown that this sequence modulo 1 has a distribution function.

2. Auxiliary results. In the first part of this section we establish three lemmas, the first one is well-known and due to A. Baker [1]. The other ones are completely elementary.

Lemma 1. *Let*

$$\lambda=h_1\cdot\log y_1+\cdots+h_s\cdot\log y_s,$$

where h_i's are rational integers and y_i's denote algebraic numbers ($y_i\neq 0$ or 1). We assume that not all of the h_i's are 0 and that the logarithms mean their principal values. Suppose that $\max(|h_i|)\leq B$ (≥ 4), *y_i has height (the maximum of the absolute values of the coefficients in its defining polynomial) at most M_i* (≥ 4) *and that the field generated by the y_i's over the rational numbers has degree at most d. If $\lambda\neq 0$, then*

$$|\lambda|>B^{-c_0\Omega\cdot\log\Omega'},$$

where

$$\Omega=\log M_1\cdot\log M_2\cdots\log M_s,$$
$$\Omega'=\Omega/\log M_s$$

and c_0 *is an effectively computable constant depending only on s and d.*

Lemma 2. *Let z and w be non-real complex numbers for which* $zw \neq 1$. *Then there is a real number* $c_1 > 0$, *which depend only on z and* $|w|$, *such that*

$$\left|1 - \left|\frac{\bar{z}w-1}{zw-1}\right|\right| \geqq \min\{1, c_1 \cdot |\mathrm{Im}(w)|\},$$

where $\bar{z}$ *denotes the complex conjugate of z.*

Proof. Let $z = z_1 + z_2 i$, $w = w_1 + w_2 i$ and $\max(|z|, |w|) = V$. By some elementary argument we get

$$\begin{aligned} \left|\frac{\bar{z}w-1}{zw-1}\right| &= \left|\frac{(z_1w_1+z_2w_2-1)+(z_1w_2-z_2w_1)i}{(z_1w_1-z_2w_2-1)+(z_1w_2+z_2w_1)i}\right| \\ &= \sqrt{\frac{(z_1w_1+z_2w_2-1)^2+(z_1w_2-z_2w_1)^2}{(z_1w_1-z_2w_2-1)^2+(z_1w_2+z_2w_1)^2}} \\ &= \sqrt{1-\frac{4z_2w_2}{(z_1w_1-z_2w_2-1)^2+(z_1w_2+z_2w_1)^2}}. \end{aligned} \tag{6}$$

But by $\max(|z_1|, |z_2|, |w_1|, |w_2|) \leqq V$

$$\left|\frac{4z_2w_2}{(z_1w_1-z_2w_2-1)^2+(z_1w_2+z_2w_1)^2}\right| \geqq \frac{4|z_2|}{(2V^2+1)^2+4V^4} \cdot |w_2| \tag{7}$$

follows, and

$$|1-\sqrt{1+\delta}| \geqq \min\left(1, \frac{|\delta|}{3}\right) \tag{8}$$

for any $\delta \geqq -1$, so by (6), (7), and (8) the lemma is proved.

Lemma 3. *Let* $y_1, \cdots, y_s$ *be a multiplicatively independent system of unimodular complex algebraic numbers, i.e.* $|y_k| = 1$ *for* $k = 1, \cdots, s$ *and* $y_1^{h_1} \cdots y_s^{h_s} \neq 1$ *for all non-zero integral s-tuples* $(h_1, \cdots, h_s)$. *Then there are positive numbers* c_2 *and* n_0 *depending only on the system* $y_1, \cdots, y_s$ *such that*

$$|1 - y_1^{h_1} \cdots y_s^{h_s}| < e^{-c_2 \cdot \log \max(|h_1|, \cdots, |h_s|)}$$

for any integral s-tuple with $\max(|h_1|, \cdots, |h_s|) > n_0$.

Proof. Since $1 - y_1^{h_1} \cdots y_s^{h_s} \neq 0$ we have

$$|1 - y_1^{h_1} \cdots y_s^{h_s}| \geqq \frac{2}{\pi} |\arg y_1^{h_1} \cdots y_s^{h_s}| = \frac{2}{\pi} |\log y_1^{h_1} \cdots y_s^{h_s}|,$$

where we have used the elementary inequalities $\sin\varphi \geqq (2/\pi)\varphi$ for $0 \leqq \varphi \leqq (\pi/2)$ and $1 - \cos\varphi \geqq (2/\pi)\varphi$ for $(\pi/2) \leqq \varphi \leqq \pi$. Thus we obtain

$$|1 - y_1^{h_1} \cdots y_s^{h_s}| \geqq \frac{2}{\pi} |h_1 \cdot \log y_1 + \cdots + h_s \cdot \log y_s - t \cdot \log(-1)|,$$

where the logarithms take their principal values and t is an integer with $|t| < 2(|h_1| + \cdots + |h_s|)$. Using Lemma 1 for $m = \max(|h_1|, \cdots, |h_s|) > n_0$ we get

$$|1 - y_1^{h_1} \cdots y_s^{h_s}| \geqq \frac{2}{\pi} (2sm)^{-c_2'} > e^{-c_2 \log m},$$

and the proof of the lemma is complete.

(to be continued.)

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