4. Invariant Spherical Distributions of Discrete Series on Real Semisimple Symmetric Spaces G_C/G_R

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For real semisimple connected Lie groups G_R, Harish-Chandra discussed in [2] invariant eigendistributions on the groups corresponding to the characters of discrete series. In this paper, we study invariant spherical distributions (=ISD's) of discrete series for the symmetric spaces G_C/G_R and the unitary representations associated to the ISD's, for the complexification G_C of G_R. In [6] and [7, 8], the cases of $SL(2,C)/SL(2,R)$, $Sp(2,C)/Sp(2,R)$ and $GL(n,C)/GL(n,R)$ were treated, where the discrete series appears. In [5] and [9], we discussed general theories for the symmetric spaces G_C/G_R. From these works, we can see that there exists an interesting duality between the series of ISD's on G_C/G_R and those of invariant eigendistributions on G_R in such a way that the discrete series corresponds to the continuous series and vise versa.

§1. Invariant spherical distributions of discrete series for G_C/G_R. Assume that G_R has a simply connected complexification G_C. Let σ be an involutive automorphism of G_C such that $(G_C)^\sigma=G_R$, where $(G_C)^\sigma$ is the set of all fixed points of σ in G_C. Put $X=\{g\sigma(g)^{-1}: g\in G_C\}$, then G_C/G_R and X are isomorphic under $G_C/G_R\ni gG_R\mapsto g\sigma(g)^{-1}\in X$ as G_C-spaces. Let $\mathfrak{g}_R$ be the Lie algebra of G_R and $\mathfrak{g}_C$ its complification.

We assume throughout this paper that the symmetric pair $(\mathfrak{g}_C, \mathfrak{g}_R)$ admits a compact Cartan subspace $\mathfrak{b}$. In this case, there exists the discrete series for X. Any root of $(\mathfrak{g}_C, \mathfrak{b}_C)$ is singular imaginary with respect to $\mathfrak{g}_R$ (cf. [10, p. 509]). Let $\mathfrak{a}_1=\mathfrak{b}, \mathfrak{a}_2, \cdots, \mathfrak{a}_n$ be a maximal set of Cartan subspaces of $(\mathfrak{g}_C, \mathfrak{g}_R)$, not G_R-conjugate each other. Recall that $X\subset G_C$ and put $A_i=Z_X(\mathfrak{a}_i)$ and $W^i=N_{G_R}(A_i)/Z_{G_R}(A_i)$ for $1\le i\le n$. Consider the polynomial in t: $\det((1+t)\,\mathrm{Id}\text{-}\mathrm{Ad}(x))=\sum_{i=0}^m t^iD_i(x)$, $m=\dim\mathfrak{g}_C$. Let l be the smallest integer such that $D_l(x)\not\equiv 0$. The set X' of regular elements in X is an open dense subset of X and $X'=\bigcup_{i=1}^n G_R[A_i']$ with $A_i'=A_i\cap X$ and $G_R[A_i']=\bigcup_{g\in G_R} gA_i'g^{-1}$. Since $\mathfrak{a}_1$ is compact, the subspace A_1 of X is an abelian connected group. Let A_1^* be the unitary character group of A_1, then it can be identified with a lattice F in the dual space of $\sqrt{-1}\,\mathfrak{b}$: for $\lambda\in F$, there exists a unique element a^* of A_1^* such that $\langle a^*, \exp H\rangle=e^{\lambda(H)}$ $(H\in\mathfrak{b})$. Let W be the Weyl group of $(\mathfrak{g}_C, \mathfrak{b}_C)$. For any $w\in W$, there exists an element $\underline{w}\in W^1$ such that $e^{w\lambda(H)}=\langle a^*, \underline{w}(\exp H)\rangle$ for $H\in\mathfrak{b}$. An element $\lambda\in F$ is called regular if $w\lambda\neq\lambda$ for any $w\in W$, $\neq 1$, and the set of all regular elements of F will be denoted by F'. Denote by $D(X)$ the algebra of G_C-

invariant differential operators on $\boldsymbol{X}$. Let $\gamma^{\mathfrak{b}}$ be an isomorphism given in [5, § 3] of $\boldsymbol{D}(\boldsymbol{X})$ onto $I(\mathfrak{b})$, the set of W-invariant elements of the universal enveloping algebra $U(\mathfrak{b}_C)$ of $\mathfrak{b}_C$. For $\lambda \in \boldsymbol{F}$, let χ_λ be the homomorphism of $\boldsymbol{D}(\boldsymbol{X})$ into $\boldsymbol{C}$ given by $\chi_\lambda(Z)=\gamma^{\mathfrak{b}}(Z)(\lambda)$ $(Z \in \boldsymbol{D}(\boldsymbol{X}))$. From Theorem 5.1 in [9], we obtain

Theorem 1. *Fix an element λ of F. There exists an ISD Θ_λ satisfying the following conditions:*

(i) $Z\Theta_\lambda=\chi_\lambda(Z)\Theta_\lambda$ *for any* $Z \in \boldsymbol{D}(\boldsymbol{X})$,

(ii) $\sup\{|D_\iota(x)|^{1/4}|\Theta_\lambda(x)| : x \in \boldsymbol{X}'\}<\infty$,

(iii) $\Theta_\lambda(a)=\{\sum_{w\in W} e^{w\lambda(\log a)}\}|D_\iota(a)|^{-1/4}$ $(a \in A_1')$.

In case $\lambda \in \boldsymbol{F}'$, Θ_λ is determined uniquely and the support of Θ_λ is contained in the closure of $G_R[A_1']$. For general $\lambda \in \boldsymbol{F}$, the G_R-invariant analytic function $\Theta_\lambda(a)$ on $G_R[A_1']$ given by (iii) *determines an ISD on $\boldsymbol{X}$ by $\Theta_\lambda(f)=\int_{G_R[A_1']} f(x)\Theta_\lambda(x)dx$ $(f \in C_c^\infty(\boldsymbol{X}))$, where dx is a G_C-invariant measure on $\boldsymbol{X}$* ((i), (ii) *are naturally satisfied).*

§ 2. Tempered invariant spherical distributions. Let θ be a Cartan involution of $\mathfrak{g}_C$ commuting with σ and $\mathfrak{g}_C=\mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition. Denote by $B(\ )$ the Killing form of $\mathfrak{g}_C$. Put $\mathfrak{g}_R^a=(\mathfrak{g}\cap\mathfrak{k})+(\sqrt{-1}\,\mathfrak{g}\cap\mathfrak{p})$. We fix a maximal abelian subspace $\mathfrak{a}$ of $(\sqrt{-1}\,\mathfrak{g}_R)\cap\mathfrak{p}$ and choose a positive root system Σ^+ of $(\mathfrak{g}_R^a, \mathfrak{a})$. Put $\mathfrak{a}^+=\{X \in \mathfrak{a} : \alpha(X)\geq 0$ for all $\alpha \in \Sigma^+\}$ and $A^+=\exp\mathfrak{a}^+$. Let K be the analytic subgroup of G_C corresponding to $\mathfrak{k}$, then $G_C=KA^+G_R$. For any $g \in G_C$, there exists a unique element X of $\mathfrak{a}^+$ such that $g \in K(\exp X)G_R$. Then, for $x=g\sigma(g)^{-1} \in \boldsymbol{X}$, we define functions $\tau(x)$ and $\Xi(x)$, on $\boldsymbol{X}$ by $\tau(x)=-B(X,\theta(X))$, $\Xi(x)=e^{\rho(X)}$, where $\rho=(1/2)\sum_{\alpha>0} m_\alpha\cdot\alpha$, $m_\alpha=\dim\mathfrak{g}_C(a:\alpha)$. For $f \in C(\boldsymbol{X})$, put

$$\nu_r(f)=\sup_{x\in X}(1+\tau(x))^r\Xi(x)^{-1}|f(x)| \qquad (r \in \boldsymbol{R}).$$

For $X \in \mathfrak{g}$, we associate a differential operator on $\boldsymbol{X}$ as

$$f(X;x)=\frac{d}{dt}f(\exp(tX)\cdot x\cdot\sigma(\exp tX)^{-1})|_{t=0}(f \in C^\infty(\boldsymbol{X})).$$

Extend this correspondence to $U(\mathfrak{g}_C)$ and put $\nu_{r,D}(f)=\nu_r(f(D;x))$ $(D \in U(\mathfrak{g}_C))$. We define the space of rapidly decreasing functions on $\boldsymbol{X}$ by

$$\mathcal{S}(\boldsymbol{X})=\{f \in C^\infty(\boldsymbol{X}) : \nu_{r,D}(f)<\infty \text{ for } r \in \boldsymbol{R} \text{ and } D \in U(\mathfrak{g}_C)\}.$$

A distribution on $\boldsymbol{X}$ is called tempered if it can be extended continuously to $\mathcal{S}(\boldsymbol{X})$.

Theorem 2. *The ISD Θ_λ $(\lambda \in F)$ given in Theorem 1 is tempered.*

For a finite dimensional irreducible representation δ of K, let ξ_δ denote its character and $d(\delta)$ its degree. Put

$$(\delta * f)(x)=d(\delta)\int_K \xi_\delta(k^{-1})f(k^{-1}x\sigma(k))dk \qquad (f \in C(\boldsymbol{X}))$$

where dk is a Haar measure on K. Let δ^* be the contragredient representation of δ, and for any distribution Θ on $\boldsymbol{X}$, define a distribution Θ_δ by $\Theta_\delta(f)=\Theta(\delta * f)$ $(f \in C_c^\infty(\boldsymbol{X}))$.

Theorem 3. *Let Θ_λ be the ISD given in Theorem* 1, *then* $\Theta_{\lambda,\delta} \in L^2(X)$ *for* $\lambda \in F'$.

Let $\mathcal{R}$ be the representation of G_C on $L^2(X)$ defined by $[\mathcal{R}_g f](x) = f(g^{-1}x\sigma(g))$ $(f \in L^2(X))$. Let V^λ denote the $\mathcal{R}$-stable minimal closed subspace of $L^2(X)$ spanned by $\Theta_{\lambda,\delta}$. The restriction of $\mathcal{R}$ to V^λ is denoted by T^λ. We can prove that T^λ is irreducible.

§3. Representations of discrete series for G_C/G_R. Let B_0 be the analytic subgroup of G_C corresponding to $\mathfrak{b}_0 = \sqrt{-1}\,\mathfrak{b}$ and M the centralizer of B_0 in K, then $M = A_1$. For $\lambda \in F$, there exists an irreducible representation σ_λ of M such that $\sigma_\lambda(\exp H) = e^{\lambda(H)}$ $(H \in \mathfrak{b})$. MB_0 is a Cartan subgroup of G_C. Let $P = MB_0N$ be a Borel subgroup of G_C. For a complex valued linear form μ on $\mathfrak{b}_0$, let ξ_μ be the character of B_0 defined by $\xi_\mu(\exp X) = e^{\mu(X)}$ $(X \in \mathfrak{b}_0)$. The unitary representation of G_C induced from the representation $\sigma_\lambda \otimes \xi_\mu \otimes 1$ of P is denoted by $\pi_{\lambda,\mu}$.

Theorem 4. *For any* $\lambda \in F'$, *the irreducible unitary representation* T^λ *of* G *is equivalent to* $\pi_{2\lambda,0}$.

For the symmetric spaces $Sp(2, C)/Sp(2, R)$ and $GL(n, C)/GL(n, R)$ in [7, 8], these ISD's cover exactly the discrete part of the Fourier inversion formula. In general, they correspond to the discrete series given in [4].

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