1. Nonlinear Eigenvalue Problem $\Delta u+\lambda e^{u}=0$ on Simply Connected Domains in R^{2}

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§ 1. **Introduction and results.** In the previous work [3, 4], we studied the connectivity of the branch of minimal solutions $\underline{C}$ starting from $(\lambda, u)=(0, 0)$ and that of Weston-Moseley's large solutions C^{*} as $\lambda \downarrow 0$ ([6, 2]) for the nonlinear eigenvalue problem

(1.1) $$-\Delta u=\lambda e^{u} \text{ (in } \Omega) \quad \text{and} \quad u=0 \text{ (on } \partial\Omega),$$

where λ is a positive constant, $\Omega \subset R^{2}$ is a simply-connected bounded domain with smooth boundary $\partial\Omega$, and $u \in C^{2}(\Omega) \cap C^{0}(\bar{\Omega})$ is a classical solution. We have established the connectivity of $\underline{C}$ and C^{*} when Ω is close to a disc. In this note, we shall refine the result and give an explicit criterion for Ω to have such a property for (1.1).

Our basic idea was to parametrize the solutions $h={}^{T}(u, \lambda)$ of (1.1) through $s=\lambda\int_{\Omega} e^{u}dx$. Thus we introduce the nonlinear mapping $\Phi=\Phi(h, s)$: $\hat{X}\times R \rightarrow \hat{Y}$ by $\Phi(h, s)={}^{T}(\Delta u+\lambda e^{u}, \int_{\Omega} e^{u}dx-(s/\lambda))$ for $h={}^{T}(u, \lambda)$ and $s \in R_{+}$, where $\hat{X}={}^{T}(X\times R_{+})$ and $\hat{Y}={}^{T}(Y\times R)$ with $X=C_{0}^{2+\alpha}(\bar{\Omega})\equiv\{v \in C^{2+\alpha}(\bar{\Omega})| v=0 \text{ on } \partial\Omega\}$ and $Y=C^{\alpha}(\bar{\Omega})$ for $0<\alpha<1$. For this mapping we claim that

Theorem 1. *For each zero-point (h, s) of Φ, the linearized operator $d_{h}\Phi(h, s)$: $\hat{X}\rightarrow\hat{Y}$ is invertible provided that $0<s<8\pi$.*

Since the a priori estimates $\|u\|_{C^{0}(\bar{\Omega})}\leqq -2\log(1-(s/8\pi))$ and $s|\Omega|^{-1}\exp(\|u\|_{C^{0}(\bar{\Omega})})\leqq\lambda\leqq\bar{\lambda}$ hold if $0<s<8\pi$ for some $\bar{\lambda}=\bar{\lambda}(\Omega)$, the first part of the following theorem follows immediately from the above one. On the other hand the latter part holds by the fact that $s<4\pi$ and $s<8\pi$ imply $\mu_{1}(p)>0$ and $\mu_{2}(p)>0$, respectively, where $\{\mu_{j}(p)\}_{j=1}^{\infty}(-\infty<\mu_{1}(p)<\mu_{2}(p)\leqq\cdots\rightarrow\infty)$ are the eigenvalues of $A_{p}\equiv -\Delta-p$ under Dirichlet condition for $p=\lambda e^{u}$:

Theorem 2. *In $s-h$ plane, there exists a branch $\mathcal{S}$ of zero-points of Φ starting from $(s, h)=(0, 0)$ and continuing up to $s=8\pi$ without bending, and furthermore, there is no other zero-point of Φ other than $\mathcal{S}$ in the area $0<s<8\pi$. The corresponding branch C in $\lambda-u$ plane to $\mathcal{S}$ starts from $(\lambda, u)=(0, 0)$ and bends at most once.*

On the other hand, along the Weston-Moselely's branch C^{*} of large solutions, we have from [4] that $S\equiv\lambda\int_{\Omega} e^{u}dx=8\pi+c\pi\lambda+0(\lambda)$ as $\lambda \downarrow 0$, where $C=C(\Omega)=-|a_{1}|^{2}+\sum_{n=3}^{\infty}(n^{2}/(n-2))|a_{n}|^{2}$ for the normalized Riemann mapping

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$g=g_N: D=\{|z|<1\}\to\Omega$ with $g(z)=\sum_{n=0}^{\infty} a_n z^n$ $(a_2=0)$. Thus we obtain

Theorem 3. *In the case of $C<0$, Weston-Moseley's branch C^* exists uniquely and connects with the minimal branch $\underline{C}$. The connected branch C bends just once in $\lambda-u$ plane.*

In fact $C<0$ implies $\alpha\equiv|g_N''(0)/g_N'(0)|<2$, in which case the constrains for Ω to produce C^* are all verified by the method of Wente [5]. We know that $C<0$ holds if $\kappa|g'|<2$ everywhere on $\partial\Omega$, where κ is the curvature of $\partial\Omega$ ([4]). When Ω is a ball, we have $\kappa|g'|\equiv 1$. Incidentally, $\alpha<2$ whenever Ω is convex ([2]). In case $C>0$ or $\alpha>2$, multiple Weston-Moseley's branches may exist, of which global analysis will be a theme in future.

§2. Proof of Theorem 1. Let $\Phi(h,s)=0$ for some $h\in\hat{X}$ and $s\in(0,8\pi)$. Then, the linearized operator $d_h\Phi(h,s)$ can be regarded as a self-adjoint operator in ${}^T(L^2(\Omega)\times\boldsymbol{R})$ with the domain ${}^T(H^2\cap H_0^1(\Omega)\times\boldsymbol{R})$. The associated sesqui-linear form $\mathfrak{A}=\mathfrak{A}(\ ,\ )$ on ${}^T(H_0^1(\Omega)\times\boldsymbol{R})$ is given for $f={}^T(v,\kappa)$ and $g={}^T(w,\rho)$ that $\mathfrak{A}(f,g)=-a(\tilde{f},\tilde{g})$, where $a(v,w)=\int_\Omega\{\nabla v\cdot\nabla w-pvw\}dx$ for $p=\lambda e^u$ and $\tilde{f}=f+(\kappa/\lambda)$ and $\tilde{g}=g+(\rho/\lambda)\in\hat{V}\equiv\{v\in H^1(\Omega)\,|\,(\partial/\partial\tau)v=0 \text{ on } \partial\Omega\}$ for a unit tangential vector τ. Thus $0\in\rho(d_h\Phi(h,s))$ is equivalent to $0\in\rho(\hat{A}_p)$, where $\hat{A}_p$ is the self-adjoint operator in $L^2(\Omega)$ associated with $a|_{\hat{V}\times\hat{V}}$. See [3] for details.

Putting $\sigma(\hat{A}_p)=\{\hat{\mu}_j(p)\}_{j=1}^{\infty}$ with $-\infty<\hat{\mu}_1(p)<\hat{\mu}_2(p)\leqq\cdots$, we have $\hat{\mu}_1(p)<0$ because constant functions belong to $\hat{V}$. Furthermore, $\hat{\mu}_2(p)>0$ if $s>0$ is small. We shall extend this consequence and show that $\hat{\mu}_2(p)>0$ whenever $0<s<8\pi$. To this end, we first note that this fact holds when $\Omega=D\equiv\{|z|<1\}$. In fact, in this case all solutions are parametrized by $s=\lambda\int_\Omega e^u dx$ as $\{(\lambda^*(s),u^*(s))\,|\,0<s<8\pi\}$ with the property that $d_h\Phi(h^*(s),s)$ is invertible for $0<s<8\pi$, where $h^*(s)={}^T(u^*(s),\lambda^*(s))$ ([4]). Hence $\mu_2(p^*(s))>0$ $(0<s<8\pi)$ holds for $p^*(s)=\lambda^*(s)e^{u^*(s)}$.

Next, we note that $\hat{\mu}_2(p)>0$ is equivalent to $\hat{\nu}_2(p)>1$, where $\{\hat{\nu}_j(p)\}_{j=1}^{\infty}$ $(0=\hat{\nu}_1(p)<\hat{\nu}_2(p)\leqq\cdots\to+\infty)$ denotes the set of eigenvalues for

$$\text{(EVP)}\qquad \varphi\in\hat{V}\quad\text{and}\quad\int_\Omega\nabla\varphi\cdot\nabla\chi\,dx=\nu\int_\Omega\varphi\chi p\,dx\quad\text{for any }\chi\in\hat{V}.$$

The first eigenfunction corresponding to $\hat{\nu}_1(p)=0$ for (EVP) is a constant, so that we have $\hat{\nu}_2(p)=\mathrm{Inf}\left\{R(v)\,|\,v\in\hat{V},\ \int_\Omega vp\,dx=0\right\}$ by mini-max principle, where $R(v)=\int_\Omega|\nabla v|^2dx\Big/\int_\Omega v^2p\,dx$. Minimizer φ of this variational problem is a second eigenfunction and hence is analytic in Ω and has two nodal domains $\Omega_\pm$ in Ω. At least one of $\Omega_\pm$ meets $\partial\Omega$. Without loss of generality, we suppose $\partial\Omega_-\cap\partial\Omega\neq\emptyset$ and put $\varphi_\pm=(\pm\varphi)\vee 0$. Here we take generalized Schwarz' symmetrization $\varphi_-^*\in\hat{V}^*$ of φ_- ([1]) in use of the cannonical radial metric p^*ds^2 on D giving $1/2$ Gaussian curvature and $s=\int_\Omega p\,dx=\int_D p^*dx$, where $\hat{V}^*=\{v\in H^1(D)\,|\,(\partial/\partial\tau)v=0 \text{ on } \partial D\}$. Namely, $\varphi_-^*(x)=\sup\{\mu\,|\,x\in D_\mu^*\}$, where D_μ^* is the concentric disc in D such that $\int_{D_\mu^*} p^* dx = \int_{D\mu} p dx$ for $D_\mu = \{x \mid \varphi_-(x) < \mu\}$. Then, we have $\int_\Omega |\nabla \varphi_-|^2 dx \geqq \int_D |\nabla \varphi_-^*|^2 dx$ by $\varphi_-|_{\partial\Omega} = 0$ as well as $\int_\Omega \varphi_- p dx = \int_D \varphi_-^* p^* dx$ and $\int_\Omega \varphi_-^2 p dx = \int_D \varphi_-^{*2} p^* dx$ ([1]). On the other hand, for φ_+ we take $\varphi_{+*} \in \hat{V}^*$ as $\varphi_{+*}(x) = \mathrm{Inf}\{\mu \mid x \in A_\mu^*\}$, where A_μ^* is the concentric annulus in D such that $\partial D \subset \partial A_\mu^*$ and $\int_{A\mu} p^* dx = \int_{A\mu} p dx$ for $A_\mu = \{x \mid \varphi_+(x) > \mu\}$. Then, similar properties hold for this rearrangement[1]. That is, $\int_\Omega \varphi_+ p dx = \int_D \varphi_{+*} p^* dx$, $\int_\Omega \varphi_-^2 p dx = \int_D \varphi_{+*}^2 p^* dx$ and $\int_\Omega |\nabla \varphi_+|^2 dx \geqq \int_D |\nabla \varphi_{+*}|^2 dx$. Furthermore, $\operatorname{supp} \varphi_-^* \cap \operatorname{supp} \varphi_{+*}$ is just a circle so that we have for $\varphi^* = \varphi_{+*} - \varphi_-^* \in \hat{V}^*$ that $\int_D \varphi^* p^* dx = 0$, $\int_D \varphi^{*2} p^* dx = \int_\Omega \varphi^2 p dx$ and $\int_D |\nabla \varphi^{*2}| dx \leqq \int_\Omega |\nabla \varphi|^2 dx$ and hence we obtain $\nu_2(p) \geqq \nu_2(p^*) = \mathrm{Inf}\left\{R^*(v) \mid v \in \hat{V}^*, \int_D v p^* dx = 0\right\}$, where $R^*(v) = \int_D |\nabla v|^2 dx \Big/ \int_D v^2 p^* dx$. However $p^* = \lambda^*(s) e^{u^*(s)}$, where $h^*(s) = {}^T(u^*(s), \lambda^*(s))$ is the radial solution of (1.1) for $\Omega = D$ with $s = \int_\Omega p^* dx$, so that $\nu_2(p^*) > 1$.

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[1] Note that φ is constant on $\partial\Omega$.