19. On the Representation of the Scattering Kernel for the Elastic Wave Equation

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Introduction. In Yamamoto [7] and Shibata and Soga [4] we have known that we can construct the scattering theory for the elastic wave equation corresponding to the theory for the scalar-valued wave equation formulated by Lax and Phillips [1, 2]. On Lax and Phillips' formulation Majda [3] obtained a representation of the scattering kernel (operator), which is very useful for consideration on the inverse scattering problems (cf. Majda [3], Soga [5, 6], etc.). In the present note we shall give the similar representation of the scattering kernel for the elastic wave equation considered in Shibata and Soga [4].

§1. Main results. Let Ω be an exterior domain in $\boldsymbol{R}^n_x$ $(x=(x_1, \cdots, x_n))$ whose boundary $\partial\Omega$ is a compact C^∞ hypersurface. Throughout this note we assume that the dimension n is odd and $\geqq 3$. Let us consider the elastic wave equation

$$(1.1)\qquad \begin{cases} \left(\partial_t^2 - \sum_{i,j=1}^n a_{ij}\partial_{x_i}\partial_{x_j}\right)u(t,x)=0 & \text{in } \boldsymbol{R}\times\Omega, \\ Bu(t,x)=0 & \text{on } \boldsymbol{R}\times\partial\Omega, \\ u(0,x)=f_1(x), \quad \partial_t u(0,x)=f_2(x) & \text{on } \Omega. \end{cases}$$

Here, a_{ij} are constant $n\times n$ matrices whose (p, q)-component a_{ipjq} satisfies

(A.1) $\quad a_{ipjq}=a_{ptjq}=a_{jqip}, \qquad i, j, p, q=1, 2, \cdots, n,$

(A.2) $\quad \sum_{i,p,j,q=1}^n a_{ipjq}\varepsilon_{jq}\bar{\varepsilon}_{ip} \geqq \delta \sum_{i,p=1}^n |\varepsilon_{ip}|^2 \qquad$ for Hermitian matrices (ε_{ij}),

(A.3) $\quad \sum_{i,j=1}^n a_{ij}\xi_i\xi_j$ has characteristic roots of constant multiplicity for $\xi=(\xi_1, \cdots, \xi_n)\in \boldsymbol{R}^n-\{0\}$,

and the boundary operator B is of the form

$$Bu=u|_{\partial\Omega} \quad \text{or} \quad \sum_{i,j=1}^n \nu_i(x)a_{ij}\partial_{x_j}u|_{\partial\Omega},$$

where $\nu=(\nu_1, \cdots, \nu_n)$ is the unite outer vector normal to $\partial\Omega$. We denote by $U(t)$ the mapping: $f=(f_1, f_2)\to(u(t, \cdot), \partial_t u(t, \cdot))$ associated with (1.1), and by $U_0(t)$ the one associated with the equation in the free space ($\Omega=\boldsymbol{R}^n$).

Under the assumptions (A.1)–(A.3) it has been proved in Shibata and Soga [4] that the wave operators $W_\pm=\lim_{t\to\pm\infty} U(-t)U_0(t)$ are well defined and complete (cf. §3 of [4]). Let $\{\lambda_j(\xi)\}_{j=1,\dots,d}$ $(\lambda_1<\cdots<\lambda_d)$ be the eigenvalues of $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j$, and let $P_j(\xi)$ be the projection into the eigenspace of $\lambda_j(\xi)$. For the data $f=(f_1, f_2)$ $(\in \mathcal{S})$ in the free space, let us set

$$T_0 f(s,\omega)=\sum_{j=1}^d \lambda_j(\omega)^{1/4}P_j(\omega)(-\lambda_j(\omega)^{1/2}\partial_s^{(n+1)/2}\tilde{f}_1+\partial_s^{(n-2)/2}\tilde{f}_2)(\lambda_j(\omega)^{1/2}s, \omega),$$

where $\tilde{f}_i(s, \omega)=\int_{x\cdot\omega=s} f_i(x)dS_x$, $(s, \omega) \in \boldsymbol{R}\times S^{n-1}$. Then T_0 becomes the translation representation for the equation in the free space (cf. §2 in Shibata and Soga [4]). We define the scattering operator S by $S=T_0W_+^{-1}W_-T_0^{-1}$, as Lax and Philips [1, 2] did. S is a unitary operator from $L^2(\boldsymbol{R}\times S^{n-1})$ to itself.

The main purpose of this note is to give a representation of S similar to Majda's in [3]. Derivation of this representation is based on the following

Theorem 1. *Let* (A.1)–(A.3) *be satisfied, and assume that*

(A.4) *every slowness hypersurface* $\Sigma_j=\{\xi : \lambda_j(\xi)=1\}$ *is strictly convex.*

Then, for any f *with* $T_0f \in \mathcal{S}(\boldsymbol{R}\times S^{n-1})$ *we have*

$$T_0f(s, \theta)=\lim_{t\to+\infty}(\pi t)^{(n-1)/2}\sum_{j=1}^{d}K_j(\theta)^{1/2}|\partial_\xi\lambda_j(\theta)|^{(n+1)/2}\lambda_j(\theta)^{-(2n+1)/4}\cdot(U_0(t)f)_2(2^{-1}\lambda_j(\theta)^{-1/2}t\partial_\xi\lambda_j(\theta)+s\lambda_j(\theta)^{1/2}\theta),$$

where $K_j(\theta)$ *denotes the Gaussian curvature of* Σ_j *at* $\lambda_j(\theta)^{-1/2}\theta$.

Let $v_l(t, x; \omega)$ be the solution of the equation

$$\begin{cases}\partial_t^2 v-\sum_{i,j=1}^{n}a_{ij}\partial_{x_i}\partial_{x_j}v=0 & \text{in } \boldsymbol{R}\times\Omega,\\ Bv=-2^{-1}(-2\pi i)^{1-n}\lambda_l(\omega)^{-n/4}B\{\delta(t-\lambda_l(\omega)^{-1/2}\omega\cdot x)P_l(\omega)\} & \text{on } \boldsymbol{R}\times\partial\Omega,\\ v=0 \quad \text{if } t \text{ is small enough.}\end{cases}$$

$v_l(t, x; \omega)$ is an $n\times n$ matrix of C^∞ functions of x and ω with the value of the distribution in t.

Theorem 2. *Let us assume* (A.1)–(A.4), *and set*

$$S_0(s, \theta, \omega)=\sum_{i,j=1}^{d}\int_{\partial\Omega}\lambda_i(\theta)^{-n/4}\{P_i(\theta)(\partial_t^{n-2}Nv_j)(\lambda_i(\theta)^{-1/2}\theta\cdot x-s, x; \omega)-\lambda_i(\theta)^{-1/2}N(\theta\cdot x)P_i(\theta)(\partial_t^{n-1}v_j)(\lambda_i(\theta)^{-1/2}\theta\cdot x-s, x; \omega)\}dS_x,$$

where $N=\sum_{i,j=1}^{n}\nu_i(x)a_{ij}\partial_{x_j}$. *Then we have*

$$(Sk)(s, \theta)=\iint_{\boldsymbol{R}\times S^{n-1}}S_0(s-t, \theta, \omega)k(t, \omega)dtd\omega+k(s, \theta), \qquad k(s, \omega) \in C_0^\infty(\boldsymbol{R}\times S^{n-1}).$$

§2. Proof of Theorem 1. For the scalar-valued wave equation Lax and Phillips [1] obtained a theorem similar to Theorem 1 (see Theorem 2.4 in Chapter IV of [1]), but for the proof we need more precise analysis. A key lemma is the following

Lemma 1. *Let* η *and* ζ *be any elements in* $\boldsymbol{R}^n$ *with* $\eta\neq 0$. *Then, for any* $k(s, \omega) \in \mathcal{S}(\boldsymbol{R}\times S^{n-1})$ *we have*

$$\begin{aligned}\int_{S^{n-1}}&\partial_s^{(n-1)/2}k(t\lambda_j(\omega)^{-1/2}\omega\cdot\eta+\lambda_j(\omega)^{-1/2}\omega\cdot\zeta-t, \omega)d\omega\\ &=2(2\pi/|\eta|t)^{(n-1)/2}\{k(t\lambda_j(\omega_j^+)^{-1/2}\omega_j^+\cdot\eta+\lambda_j(\omega_j^+)^{-1/2}\omega_j^+\cdot\zeta-t, \omega_j^+)\\ &\qquad\cdot K_j(\omega_j^+)^{-1/2}|\partial_\xi\lambda_j(\omega_j^+)|^{-1}\lambda_j(\omega_j^+)^{(n+1)/2}\\ &\quad+2(-2\pi/|\eta|t)^{(n-1)/2}k(t\lambda_j(\omega_j^-)^{-1/2}\omega_j^-\cdot\eta+\lambda_j(\omega_j^-)^{-1/2}\omega_j^-\cdot\zeta-t, \omega_j^-)\\ &\qquad\cdot K_j(\omega_j^-)^{-1/2}|\partial_\xi\lambda_j(\omega_j^-)|^{-1}\lambda_j(\omega_j^-)^{(n+1)/2}\}+0(t^{-n/2}) \qquad \text{as } |t|\to\infty,\end{aligned}$$

where ω_j^+ (*resp.* ω_j^-) *denotes the point in* S^{n-1} *at which* $\lambda_j(\omega)^{-1/2}\omega\cdot\eta$ *is maximum* (*resp. minimum*).

In view of Theorem 2.1 in Shibata and Soga [4], we see that the limit in Theorem 1 is equal to the limit of

$$(1.2)\quad 2^{-n}\pi^{(1-n)/2}t^{(n-1)/2}\sum_{j,l=1}^{d}K_j(\theta)^{1/2}|\partial_\xi\lambda_j(\theta)|^{(n+1)/2}\lambda_j(\theta)^{-(2n+1)/4}\int_{S^{n-1}}\lambda_l(\omega)^{-n/4}P_l(\omega)$$
$$\cdot\partial_s^{(n-1)/2}T_0f(\lambda_l(\omega)^{-1/2}\omega\cdot 2^{-1}\lambda_j(\theta)^{-1/2}t\partial_\xi\lambda_j(\theta)+\lambda_l(\omega)^{-1/2}\omega\cdot\lambda_j(\theta)^{1/2}s\theta-t,\omega)d\omega$$

(as $|t|\to\infty$). Applying Lemma 1 to each integral in (1.2) yields that (1.2) converges to $T_0f(s,\theta)$ as $|t|\to\infty$. Thus Theorem 1 is obtained.

§ 3. Proof of Theorem 2. The methods of the proof are improvements of those in Soga [6]. Originally, the idea is due to Majda [3].

Lemma 2. *Let the data f in (1.1) satisfy $T_0W_-^{-1}f(s,\omega)\in C_0^\infty(\boldsymbol{R}\times S^{n-1})$, and set $k=T_0W_-^{-1}f$. Then we have*

$$(U(t)f)_2(x)=2^{-1}(2\pi)^{1-n}\sum_{j=1}^{d}\int_{S^{n-1}}\lambda_j(\omega)^{-n/4}P_j(\omega)\partial_s^{(n-1)/2}k(\lambda_j(\omega)^{-1/2}x\cdot\omega-t,\omega)d\omega$$
$$+\sum_{j=1}^{d}\iint_{\boldsymbol{R}\times S^{n-1}}\partial_t^{(n-1)/2}v_j(t+s,x;\omega)k(s,\omega)dsd\omega.$$

Lemma 3. *Let $v(t,x)$ be an $n\times n$ matrix of C^∞ functions of x with the value of the distribution in t and satisfy*

$$\begin{cases}\partial_t^2v-\sum_{i,j=1}^{n}a_{ij}\partial_{x_i}\partial_{x_j}v=0 & \text{in } \boldsymbol{R}\times\Omega,\\ v=0 & \text{if } t<r_1\end{cases}$$

(for some constant r_1). Set $N=\sum_{i,j=1}^{n}\nu_i(x)a_{ij}\partial_{x_j}$. Then we have

$$\lim_{\tau\to\infty}(\pi\tau)^{(n-1)/2}K_j(\theta)^{1/2}|\partial_\xi\lambda_j(\theta)|^{(n+1)/2}\lambda_j(\theta)^{-(n+1)/4}$$
$$\cdot\partial_t^{(n-1)/2}v(t+\tau,2^{-1}\lambda_j(\theta)^{-1/2}\partial_\xi\lambda_j(\theta_j)\tau+s\lambda_j(\theta)^{1/2}\theta)$$
$$=\int_{\partial\Omega}\{P_j(\theta)N\partial_t^{n-2}v(\lambda_j(\theta)^{-1/2}\theta\cdot x-s+t,x)$$
$$-\lambda_j(\theta)^{-1/2}N(\theta\cdot x)P_j(\theta)\partial_t^{n-1}v(\lambda_j(\theta)^{-1/2}\theta\cdot x-s+t,x)\}dS_x.$$

Lemmas 2 and 3 are extensions of Lemmas 1.3 and 1.4 in Soga [6] respectively. The proof of Lemma 2 is similar to that of Lemma 1.3 in [6], but Lemma 3 cannot be obtained in the same way as Lemma 1.4 in [6], the reason of which is that the forms of the fundamental solutions for the corresponding wave equations are fairly different.

Theorem 2 is derived from Theorem 1, Lemma 2 and Lemma 3 by the same procedures as Theorem 1 in [6] was derived from Proposition 1.2, Lemma 1.3 and Lemma 1.4 in [6].

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