18. *On Algebroid Solutions of Some Binomial Differential Equations in the Complex Plane*[†)]

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1. Introduction. The purpose of this paper is to investigate algebroid solutions of some binomial differential equations in the complex plane with the aid of the Nevanlinna theory of meromorphic or algebroid functions.

Let $a_0, \cdots, a_p$; $b_0, \cdots, b_q$ be entire functions without common zero and put

$$P(z, w)=\sum_{j=0}^{p} a_j w^j, \quad Q(z, w)=\sum_{k=0}^{q} b_k w^k \quad (a_p \cdot b_q \neq 0).$$

We consider the differential equation (D. E.):

$$(1) \qquad (w')^n = P(z, w)/Q(z, w),$$

where n is an integer. We suppose that this equation is irreducible over the set of meromorphic functions in $|z|<\infty$ and that the D. E. (1) has a nonconstant ν-valued algebroid solution $w=w(z)$ in $|z|<\infty$.

Definition. We say that w is admissible when $T(r, a_j/b_q)=o(T(r, w))$ $(0\leqq j\leqq p)$ and $T(r, b_k/b_q)=o(T(r, w))$ $(0\leqq k\leqq q-1)$ as $r\to\infty$, possibly outside a set of finite linear measure.

For example, when all a_j and b_k are polynomial, a transcendental algebroid solution of the D. E. (1) is admissible.

More than fifty years ago, K. Yosida ([11]) gave several results on algebroid solutions of the D. E. (1) when all a_j and b_k are polynomial. The followings are some of them.

Theorem A. *When all a_j and b_k are polynomial, w is of finite order and if w is transcendental,* $\max\{p, q+2n\}\leqq 2n\nu$.

There are generalizations of this theorem ([1], [3], [5], [8] etc.).

As a special case of a result of Y. He and X. Xiao ([3]), we have

Theorem B. *If w is admissible,* $p\leqq n+q+n\nu \limsup_{r\to\infty} \bar{N}(r, w)/T(r, w)$.

Recently, J. von Rieth ([6]) has studied the D. E. (1) based on K. Yosida's paper ([11]) and given some interesting results. The following is one of them.

Theorem C. *When all a_j and b_k are polynomial, if w is a transcendental solution with at most a finite number of poles, it must be $n+q\leqq p$.*

We note that in the case of Theorem C, it holds that $n+q=p$ according to Theorem B.

In this paper, we shall give some results on the solution of the D. E. (1)

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in connection with these theorems. We denote by E a subset of $[0, \infty)$ for which means $E<\infty$ and by K a constant. E or K does not always mean the same one whenever they will appear in the following. Further, the term "algebroid" will mean algebroid in the complex plane. We use the standard notation of the Nevanlinna theory of meromorphic functions ([2]) or algebroid functions ([7], [9], [10]).

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2. Lemmas. In this section, we shall give two lemmas for later use.

Lemma 1. *Let v be a transcendental algebroid function such that v and v' have at most a finite number of poles. Then, for some positive constants C_1 and C_2, it holds that*

$$M(r, v) \leqq C_1 + C_2 r M(r, v') \qquad (r \notin E),$$

where $M(r, v) = \max_{|z|=r} |v(z)|$ ([6]).

Lemma 2. *Let g be a transcendental entire function. Then,*

$$M(r, g') \leqq \sqrt{2}\{M(r, g)\}^2 \qquad (r \notin E) \qquad ([4]).$$

3. Theorems. Let $w=w(z)$ be a nonconstant ν-valued algebroid solution of the D. E. (1). We shall give some results on w in this section. We rewrite (1) as follows:

$$Q(z, w)(w')^n = P(z, w). \tag{2}$$

Theorem 1. I) *When $p \leqq n+q$, the poles of w are contained in the set of zeros of b_q.*

II) *When $p<n+q$,*

$$N(r, w) \leqq K N(r, 1/b_q).$$

Proof. Let c be a pole of w of order τ. Then, w can be expanded near c as follows:

$$w(z) = (z-c)^{-\tau/\lambda} R((z-c)^{1/\lambda})$$

where $1 \leqq \lambda \leqq \nu$ and $R(t)$ is a regular power series in t for which $R(0) \neq 0$.

I) Suppose that c is not a zero of b_q. Then, for $w=w(z)$ the order of pole of the left-hand side of (2) at $z=c$ is equal to $(n+q)\tau+n\lambda$ and that of the right-hand side is not greater than $p\tau$. This gives us the inequality $(n+q)\tau+n\lambda \leqq p\tau$, which reduces to

$$0 < n\lambda \leqq (p-n-q)\tau \leqq 0.$$

This is a contradiction.

II) Let s be the order of zero of b_q at $z=c$.

a) When the order of pole of $b_q w^q (w')^n$ is not equal to that of other terms of the left-hand side of (2) at $z=c$, we have

$$(n+q)\tau - s\lambda + n\lambda \leqq p\tau,$$

which reduces to $\tau \leqq (s-n)\lambda/(n+q-p)$.

b) When the order of pole of $b_q w^q (w')^n$ is equal to that of some other terms of the left-hand side of (2) at $z=c$, let $b_k w^k (w')^n$ be one of them, then we have

$$(n+q)\tau - s\lambda + n\lambda \leqq (n+k)\tau + n\lambda$$

which reduces to $\tau \leqq s\lambda/(q-k)$ as $q>k$.

From a) and b) we obtain our inequality.

Applying the method used in [6] to prove Theorem C, we can prove the following theorem.

Theorem 2. *Suppose that $p<n+q$ and that b_q is a polynomial. Then, we have the following inequality for $r \notin E$:*

$$\min\{n, n+q-p\}\log^+ M(r,w) \leqq \sum_{j=0}^{p} \log^+ M(r,a_j) + K\sum_{k=0}^{q-1}\log^+ M(r,b_k) + O(\log r)$$

Proof. If w is algebraic, there is nothing to prove. Therefore, we suppose that w is transcendental and $M(r,w) \geqq 1$ $(r \notin E)$. Let S be the set of zeros of b_q. Then, S is a finite set and the poles of w are contained in S by Theorem 1, I). As w is a solution of the D. E. (2), it satisfies

$$b_q^n\{\tilde{Q}(z,w)w'\}^n = P(z,w)Q(z,w)^{n-1}, \tag{3}$$

where $\tilde{Q}(z,w)=Q(z,w)/b_q$. Put for $w=w(z)$

$$U(z)=w^{q+1}/(q+1)+\sum_{k=0}^{q-1}(b_k/b_q)w^{k+1}/(k+1)$$

and

$$V(z)=\sum_{k=0}^{q-1}(b_k/b_q)'w^{k+1}/(k+1),$$

then

$$\tilde{Q}(z,w)w' = U'(z)-V(z) \tag{4}$$

and the poles of $U(z)$ are contained in S. Further, the poles of $U'(z)$ are also contained in S. In fact, substituting (4) into (3), we have

$$b_q^n\{U'(z)-V(z)\}^n = P(z,w)Q(z,w)^{n-1} \tag{5}$$

and suppose that $U'(z)$ has a pole at $z=c$ outside S. Then, the left-hand side of (5) has a pole at $z=c$, but the right-hand side of (5) has no pole at $z=c$, which is a contradiction.

As S is a finite set, we can apply Lemma 1 to U:

$$M(r,U) \leqq C_1 + C_2 r M(r,U') \qquad (r \notin E). \tag{6}$$

Let z_r be a point such that

$$M(r,U')=|U'(z_r)|, \quad |z_r|=r \quad (r \notin E).$$

Then,

$$\{M(r,U')-M(r,V)\}^n \leqq |U'(z_r)-V(z_r)|^n \leqq M(r, PQ^{n-1}/b_q^n). \tag{7}$$

By using Lemma 2 if necessary, we have

$$M(r,V) \leqq KM(r,w)^q\left\{\sum_{k=0}^{q-1}M(r,b_k)\right\}^2\Big/ r^d \qquad (r \notin E), \tag{8}$$

where d is the degree of b_q. Further,

$$M(r,PQ^{n-1}/b_q^n) \leqq KM(r,w)^{p+q(n-1)}\left\{\sum_{j=0}^{p}M(r,a_j)\right\}\left\{\sum_{k=0}^{q}M(r,b_k)\right\}^{n-1}\Big/ r^{nd} \tag{9}$$

and from the definition of U

$$M(r,U) \geqq M(r,w)^{q+1}/(q+1) - KM(r,w)^q\left\{\sum_{k=0}^{q}M(r,b_k)\right\}\Big/ r^d. \tag{10}$$

From (6)–(10) we obtain for $r \notin E$

$$M(r,w)^{\min\{1,(n+q-p)/n\}} \leqq K\left\{\sum_{j=0}^{p}M(r,a_j)\right\}^{1/n}\left\{\sum_{k=0}^{q}M(r,b_k)\right\}^{(n-1)/n}\Big/ r^d$$

$$+K\left[\sum_{k=0}^{q} M(r, b_k)+\left\{\sum_{k=0}^{q-1} M(r, b_k)\right\}^2\right]\Big/ r^d,$$

which reduces to our inequality by calculating $\log^+$ of the both sides.

Theorem 3. *When $p<n+q$, if all a_j and b_k are polynomial, any algebroid solution $w=w(z)$ of the D. E. (1) is algebraic.*

Proof. By Theorems 1 and 2, we obtain

$$T(r, w)=O(\log r) \qquad (r \notin E),$$

which shows that w is algebraic.

Here, we give two examples which show that Theorem 3 does not hold when $p \geqq n+q$.

Example 1. The D. E. $(w')^2=(-w^4+1)/4w^2$ has a transcendental algebroid solution $w=(\sin z)^{1/2}$ which has no pole. In this case, $p=n+q$.

Example 2. Let $f(z)$ be a Weierstrass' $\wp$-function which satisfies

$$(f')^2=4(f-e_1)(f-e_2)(f-e_3)$$

where $e_1+e_2+e_3=0$ and $e_1e_2e_3\neq 0$. Then, the algebroid function $w=w(z)$ defined by $w^2=f(z)$, which is transcendental, satisfies the D. E.

$$(w')^2=(w^2-e_1)(w^2-e_2)(w^2-e_3)/w^2.$$

In this case, $p>n+q$.

Finally, as a generalization of Gackstatter-Laine's conjecture ([1]) Theorem 3 suggests to us the following

Conjecture. *When $p<n+q$, any algebroid solution of the D. E. (1) would not be admissible.*

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