14. *Quadratic Conservatives of Linear Symplectic System*

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1. Introduction. The problem treated in this paper is explained as follows. Let G and g denote the $2N$-dimensional symplectic group $Sp(N, \boldsymbol{R})$ and its Lie algebra, respectively:

$$G=\{A \in M(2N, \boldsymbol{R}) \mid A'JA=J\}, \qquad g=\{X \in M(2N, \boldsymbol{R}) \mid X'J+JX=0\},$$

where $J=\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ and dash denotes matrix transpose. Our problem is to establish an algebraic approach to finding the quadratic form

$$(1) \qquad f_S(x)=x'Sx/2, \qquad S'=S,$$

conserved along any solution of the linear recurrence on $\boldsymbol{R}^N$

$$(2) \qquad x_{t+1}=Ax_t,$$

where A is an arbitrary element of G. The whole of conservatives given by (1) forms a Lie algebra with respect to the Poisson bracket [1]. This problem aims at finding economic conservation laws [2, 3] of a discrete economic growth model, though it seems trivial at a glance.

2. Linear space Ξ. In this preliminary section, we introduce a linear space Ξ of all matrices commuting with A, and it is proved that one of its subspaces is Lie algebra isomorphic to the whole of quadratic conservatives given by (1).

Now, (1) is conserved along any solution of (2), if and only if A and JS commute. Then, the whole of quadratic conservatives is identified with the following linear space of all coefficient matrices:

$$\Omega=\{S \in M(2N, \boldsymbol{R}) \mid [A, JS]=0,\ S'=S\},$$

where $[A, B]=AB-BA$. Ω forms a Lie algebra with respect to the bracket

$$(3) \qquad \langle S, T\rangle=SJT-TJS,$$

which is a representation of the Poisson bracket on Ω. Apart from looking into Ω directly, we introduce a linear space Ξ of all matrices that commute with A;

$$\Xi=\{L \in M(2N, \boldsymbol{R}) \mid [A, L]=0\}.$$

We define two linear mappings $\eta: \Xi \to \Xi$ and $\sigma: \Xi \to \Omega$ by

$$(4) \qquad \eta(L)=JL'J, \qquad \sigma(L)=J(L+\eta(L))/2=(JL+(JL)')/2.$$

Lemma 1. $\eta^2=id.$, $\eta(\Xi)=\Xi$.

Proof. Let $L \in \Xi$. Then, it follows from direct calculation that $\eta^2(L)=L$ and $[\eta(A), \eta(L)]=\eta([A, L])=0$. Since A is symplectic, we have $\eta(A)=-A^{-1}$ and accordingly $[A, \eta(L)]=0$, which means $\eta(\Xi)\subset\Xi$. This together with $\eta^2=id.$ leads to $\eta(\Xi)=\Xi$.

The lemma shows that η is an involution map on Ξ. Then, η has two eigenvalues ± 1 and Ξ is a direct sum of the two. That is, $\Xi=\Theta\dotplus\Phi$, where

$$\Theta=\{L\in \Xi \mid \eta(L)=L\}, \qquad \Phi=\{L\in \Xi \mid \eta(L)=-L\}.$$

The projector P from Ξ onto Θ is given by

$$P(L)=(L+\eta(L))/2.$$

The condition $\eta(L)=L$ is equivalent to $L'J+JL=0$ so that Θ, which is an intersection of g and Ξ, is a subalgebra of $sp(N, \boldsymbol{R})$. Then, P produces an element of g from any matrix commuting with A. It is to be stressed that A is an element of a Lie group and $P(L)$ is an element of a Lie algebra.

Now, Ξ is connected with Ω in the following manner.

Lemma 2. $\sigma(\Xi)=\Omega$, $\sigma^{-1}(0)=\Phi$.

Proof. We choose an arbitrary L. By definition, $\sigma(L)$ is symmetric. Moreover, since $J\sigma(L)=-(L+\eta(L))/2$, it belongs to Ξ and commutes with A. Thus, $\sigma(L)\in\Omega$. Conversely, for any $S\in\Omega$, we put $L=-JS$. Then, it is easily proved that $L\in\Xi$ and $\sigma(L)=S$. The second assertion is obvious from the definitions of σ and Φ.

Since $\Xi=\Theta\dotplus\Phi$, Lemma 2 shows that Ω is linearly isomorphic to Θ. Furthermore, it holds after slight calculation that

$$\sigma([L, M])=\langle\sigma(L), \sigma(M)\rangle, \tag{5}$$

where $\langle\ ,\ \rangle$ is given by (3), and L and M belong to Θ. Combining this and Lemma 2, we have

Theorem 3. *Θ is Lie algebra isomorphic to Ω.*

The linear mapping σ restricted on Θ gives a momentum mapping $\hat{J}$ in symmetry reduction theory of classical mechanics [4]. Furthermore, we note that

$$\sigma(P(L))=\sigma(L) \tag{6}$$

holds for any $L\in\Xi$.

3. A subspace Ξ_1 of Ξ. In this section, we study how many invariants are obtained among polynomials in A. To see this, we define a linear subspace Ξ_1 of Ξ

$$\Xi_1=\operatorname{span}\{I, A, A^{\pm1}, A^{\pm2}, \cdots\}.$$

Hereafter, we denote by ϕ_A and f_A the minimal polynomial and the eigenpolynomial of A, respectively. When ϕ_A is equal to f_A, any matrix that commutes with A is expressed as a polynomial in A, so that Ξ_1 coincides with Ξ. Almost any element in G has this property.

Now, we put

$$\Theta_1=\Xi_1\cap\Theta, \quad \Phi_1=\Xi_1\cap\Phi, \quad \Omega_1=\sigma(\Xi_1)=\sigma(\Theta_1).$$

Our interest centers in $\dim\Omega_1$ ($=\dim\Theta_1$), which is the number of linearly independent conservatives (1) obtained from polynomials in A.

Lemma 4. *Let $d=\dim\Omega_1$ and $k=\deg\phi_A$, and one of the following three cases holds good.*

(1) *If $k=2s+1$, then $d=s$.*

(2) *If $k=2s$ and $\phi_A(0)=1$, then $d=s$.*

(3) *If $k=2s$ and $\phi_A(0)=-1$, then $d=s-1$.*

Here, s is an integer.

Proof. We note that k is equal to $\dim\Xi_1$. For any integer i, we put

(7) $$B_i = A^i - A^{-i}$$
and $C_i = A^i + A^{-i}$. Then, it holds that $B_i \in \Theta_1$ and $C_i \in \Phi_1$. When k is an odd number $2s+1$, $\{B_1, \cdots, B_s, C_0, \cdots, C_s\}$ forms a basis of Ξ_1, and we have $k=s$. Next, when k is equal to $2s$, the $2s-1$ matrices $\{B_1, \cdots, B_{s-1}, C_0, \cdots, C_{s-1}\}$ are linearly independent and further either B_s or C_s is linearly independent of these. Now, since A is symplectic, its minimal polynomial ϕ_A satisfies $\phi_A(A) = \phi_A(A^{-1}) = 0$. Then, ϕ_A must take one of the following two forms:

(a) $(x^{2s}+1) + a_1(x^{2s-1}+x) + \cdots + a_{s-1}(x^{s+1}+x^{s-1}) + a_s x^s$,

(b) $(x^{2s}-1) + a_1(x^{2s-1}-x) + \cdots + a_{s-1}(x^{s+1}-x^{s-1})$.

When $\phi_A(0)=1$ and (a) holds, B_s becomes linearly independent. When $\phi_A(0) = -1$ and (b) holds, C_s does.

Next, we propose a simple scheme to construct a basis of Ω_1. If we do not know $\dim \Theta_1$ in advance, this scheme naturally produces a maximum number of linearly independent elements.

Theorem 5. *Suppose that $\{\sigma(A), \sigma(A^2), \cdots, \sigma(A^d)\}$ are linearly independent and $\sigma(A^{d+1})$ are linearly dependent on these d matrices. Then, the former d matrices form a basis of Ω_1.*

Proof. Since A^i belongs to G, we have $\sigma(A^i) = JB_i/2$, where B_i is given by (7). As is seen in the proof of Lemma 4, when $\dim \Theta_1 = d$, the set $\{B_1, \cdots, B_d\}$ forms a basis of Θ_1, and the converse is true. Since $(1/2)J\cdot : \theta_1 \to \Omega_1$ gives a linear isomorphism, the assertion is verified.

Again, we remark that for almost every element A of G, its minimal polynomial ϕ_A coincides with the eigenpolynomial f_A. In this case, any matrix commuting with A is expressed as a polynomial in A so that Ω_1 is Ω itself.

Theorem 6. *Suppose that for an element A of G, its minimal polynomial coincides with its eigenpolynomial. Then, $\{\sigma(A), \cdots, \sigma(A^N)\}$ forms a basis of Ω.*

Proof. Under the supposition, it holds that $\dim \Omega_1 = 2N$ and $\phi_A(0) = f_A(0) = \det(A) = 1$. Then, we have the conclusion from Lemma 4.

We can obtain all quadratic conservatives for $A \in G$ in the case of this theorem. Furthermore, we have

Theorem 7. *Under the same condition as in Theorem 6, the linear discrete system (2) is completely integrable* [5].

Proof. Since $P(A^i) = (A^i - A^{-i})/2$, we have $[P(A^i), P(A^j)] = 0$. Then, it follows from (5) and (6) that $\langle \sigma(A^i), \sigma(A^j) \rangle = 0$. That is, the system (2) admits N mutually commutative conservatives.

4. Remarks. We close this paper by showing the result of our scheme. Suppose that a discrete Lagrangian system [6] has a Lagrangian $L(q, v)$ quadratic homogeneous in q and v which is a forward difference of q. Then, after a slight calculation together with discrete Legendre transformation [6], $f_{\sigma(A)}$ (multiplied by a constant) is expressed in terms of L as

$$-\left(v'\cdot\frac{\partial L}{\partial v}-L\right)+\frac{1}{2}v'\cdot\frac{\partial L}{\partial q}.$$

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