110. A Poincaré-Birkhoff-Witt Theorem for the Quantum Group of Type A_N

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Introduction. To each complex semisimple Lie algebra $\mathcal{G}$, Jimbo [4] and Drinfeld [1, 2] associated a Hopf algebra $U_q\mathcal{G}$ with a nonzero complex parameter q. This Hopf algebra, which is called a *quantum group* by Drinfeld [2], can be considered as a natural q-analogue of the universal enveloping algebra $U\mathcal{G}$ of $\mathcal{G}$. In this note, we give an explicit linear basis of $U_q\mathcal{G}$ when $\mathcal{G}=sl_{N+1}(\boldsymbol{C})$. This result can be considered as a natural q-analogue of the Poincaré-Birkhoff-Witt theorem for $U_q sl_{N+1}(\boldsymbol{C})$. As a corollary of this, for $q(q^8-1)\neq 0$, we can show that $U_q sl_{N+1}(\boldsymbol{C})$ is a left (right) Noetherian ring, and that $U_q sl_{N+1}(\boldsymbol{C})$ has no zero divisors $\neq 0$. We also give a triangular decomposition of a general quantum group $U_q\mathcal{G}$. This is used in proving our theorem. Details which are omitted here will be published elsewhere.

1. Let F be a field and $F^\times$ the set of nonzero elements of F. Let $(a_{ij})_{1\le i,j\le N}$ be the Cartan matrix of type A_N. For $q\in F^\times$ such that $q^4\neq 1$, let $U_q sl_{N+1}$ be the associative F-algebra with 1 with generators $e_i, f_i, k_i^{\pm 1}$, $1\le i\le N$, and relations

$$k_i k_i^{-1}=k_i^{-1}k_i=1, \qquad k_i k_j=k_j k_i \tag{1.1}$$

$$k_i e_j k_i^{-1}=q^{a_{ij}}e_j, \qquad k_i f_j k_i^{-1}=q^{-a_{ij}}f_j \tag{1.2}$$

$$e_i f_j - f_j e_i=\delta_{ij}\frac{k_i^2-k_i^{-2}}{q^2-q^{-2}} \tag{1.3}$$

$$\begin{aligned} &e_i^2 e_j-(q^2+q^{-2})e_i e_j e_i+e_j e_i^2=0 && \text{for } |i-j|=1,\\ &e_i e_j-e_j e_i=0 && \text{for } |i-j|\ge 2 \end{aligned} \tag{1.4}$$

$$\begin{aligned} &f_i^2 f_j-(q^2+q^{-2})f_i f_j f_i+f_j f_i^2=0 && \text{for } |i-j|=1,\\ &f_i f_j-f_j f_i=0 && \text{for } |i-j|\ge 2. \end{aligned} \tag{1.5}$$

For $1\le i<j\le N+1$, we define inductively the elements e_{ij}, f_{ij} of $U_q sl_{N+1}$ by

$$e_{ii+1}=e_i, \qquad f_{ii+1}=f_i,$$

$$e_{ij}=qe_{ij-1}e_{j-1j}-q^{-1}e_{j-1j}e_{ij-1} \qquad \text{for } j-i\ge 2,$$

$$f_{ij}=qf_{ij-1}f_{j-1j}-q^{-1}f_{j-1j}f_{ij-1} \qquad \text{for } j-i\ge 2.$$

(The elements e_{ij}, f_{ij} were first introduced by Izumi [3], and Jimbo independently.)

Let $\Lambda_N=\{(i,j)\in \boldsymbol{Z}\times\boldsymbol{Z}\,|\,1\le i,j\le N+1\}$. Define the lexicographic order $<$ on Λ_N by

$$(i,j)<(m,n) \quad \text{if and only if} \quad i<m \quad \text{or} \quad i=m,\ j<n.$$

Now we can state our theorem.

Theorem. *Let $q\in F^\times$ such that $q^8\neq 1$. Then the elements* $f_{m_1n_1}\cdots$

$f_{m_s n_s} k_1^{l_1} \cdots k_N^{l_N} e_{i_1 j_1} \cdots e_{i_t j_t}$, $l_1, \cdots, l_1 \in \boldsymbol{Z}$, $(m_1, n_1) \leq \cdots \leq (m_s, n_s)$, $(i_1, j_1) \leq \cdots \leq (i_t, j_t)$, *form a basis of* $U_q sl_{N+1}$.

Remark. We can also give an explicit basis of $U_q sl_{N+1}$ where q is a primitive eighth root of unity.

By defining a certain filtration on $U_q sl_{N+1}$, we get the following:

Corollary. *If* $q(q^8-1) \neq 0$, *then* $U_q sl_{N+1}$ *is a left (right) Noetherian ring, and has no zero divisors* $\neq 0$.

2. Here we give a triangular decomposition of any quantum group, which is needed in proving our theorem. Let $A=(a_{ij})_{1 \leq i, j \leq N}$ be a symmetrizable generalized Cartan matrix (see [5]). Then there exist nonzero integers d_i, $1 \leq i \leq N$, such that $d_i a_{ij} = d_j a_{ji}$. For $q \in F^{\times}$ such that $q^{4d_i} \neq 1$, let $U_q \mathcal{G}_A$ be the quantum group associated with A, i.e., $U_q \mathcal{G}_A$ is the associative F-algebra with 1 with generators $e_i, f_i, k_i^{\pm 1}$, $1 \leq i \leq N$, and relations (1.1.1), (1.1.2), (1.1.3), (1.1.4), (1.1.5) in [6]. N_q^+ (resp. N_q^-) be the subalgebra of $U_q \mathcal{G}_A$ generated by $1, e_1, \cdots, e_N$ (resp. $1, f_1, \cdots, f_N$). Let H_q be the subalgebra of $U_q \mathcal{G}_A$ generated by $k_1^{\pm 1}, \cdots, k_N^{\pm 1}$.

Proposition 1. *$U_q \mathcal{G}_A$ is isomorphic to $N_q^- \otimes_F H_q \otimes_F N_q^+$ as vector spaces. The elements $k_1^{l_1}, \cdots, k_N^{l_N}$, $l_1, \cdots, l_N \in \boldsymbol{Z}$, form a basis of H_q. N_q^+ (resp. N_q^-) is characterized as the F-algebra with 1 with generators e_i (resp. f_i), $1 \leq i \leq N$, and relations* (1.1.4) (*resp.* (1.1.5)) *in* [6].

Remark. This proposition is an extention of Proposition 2 of [7].

The following proposition is obtained as an immediate consequence of Proposition 1.

Proposition 2. *For $1 \leq M \leq N$, let $A' = (a_{ij})_{1 \leq i, j \leq M}$ be the submatrix of A. Then the F-subalgebra of $U_q \mathcal{G}_A$ generated by $e_i, f_i, k_i^{\pm 1}$, $1 \leq i \leq M$, is isomorphic to $U_q \mathcal{G}_{A'}$ (as Hopf algebras).*

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