106. *Spectral Resolution of a Certain Summation of Series*

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1. Introduction. This paper deals with the spectral resolution of a certain summation of series, the final aim being to give a method of solving recurrences involving the summation by means of its spectral decomposition. Let L denote a real linear space composed of all sequences of real numbers, and a small letter, for example, a is used to mean its element $\{a_1, a_2, \cdots\}$ $(a_i \in R)$. Our summation T_d is a linear transformation on L defined by

$$(1) \qquad T_d : a \longmapsto b, \qquad b_i = \frac{1}{d^i} \sum_{j=1}^{i} \binom{i}{j} (d-1)^{i-j} a_j \quad (i=1, 2, \cdots),$$

where d is a positive number. This summation of series is closely related to the Euler summation [1].

2. Spectral resolution of T_d. In this section, we prove that $\{T_d\}_{d>0}$ is a representation of a multiplicative group, and derive the spectral resolution with the use of its group property. Let us start by showing a lemma.

Lemma 1. *Let d_1, d_2 and d be positive numbers, and we have*

$$T_{d_1} \circ T_{d_2} = T_{d_1 d_2}, \quad T_1 = I, \quad (T_d)^{-1} = T_{1/d}.$$

Proof. Suppose that

$$b_i = \frac{1}{d_2^i} \sum_{j=1}^{i} \binom{i}{j} (d_2-1)^{i-j} a_j \quad \text{and} \quad c_k = \frac{1}{d_1^k} \sum_{i=1}^{k} \binom{k}{i} (d_1-1)^{k-i} b_i.$$

Then, a slight calculation leads to

$$c_k = \frac{1}{(d_1 d_2)^k} \sum_{j=1}^{k} \binom{k}{j} (d_1 d_2 - 1)^{k-j} a_j.$$

which proves $T_{d_1} \circ T_{d_2} = T_{d_1 d_2}$. The remaining two are obvious.

This lemma shows that each T_d is a non-singular transformation and further the family $\{T_d\}_{d>0}$ is a representation on L of a Lie group (R^+, x). Exchange the parameter d for t subject to $d = e^t$ and calculate $d/dt(T_d[a])|_{t=0}$ formally. Then, we have the formal generating operator of T_d as follows;

$$(2) \qquad -a_1 \frac{\partial}{\partial a_1} + (2a_1 - 2a_2) \frac{\partial}{\partial a_2} + \cdots + (na_{n-1} - na_n) \frac{\partial}{\partial a_n} + \cdots.$$

For the time being, discussion is made on an m-dimensional linear space $\bar{L}$ which is of the first m terms $\bar{a} = \{a_1, \cdots, a_m\}$ of every element of L. It is easy to see from the definition (1) that the action of T_d can be restricted on $\bar{L}$, whose restriction we denote by $\bar{T}_d$. Then, $\bar{T}_d$ gives an R^+-action on $\bar{L}$ and its generator is expressed as a sum of first m components of (2). Since $\bar{T}_d$ is a linear transformation, it is expressed as an m-th order matrix, which is obtained by means of the generator as follows:

$$\exp\left\{t\begin{bmatrix}-1 & & & \\ 2 & -2 & & \\ & \ddots & \ddots & \\ & & m & -m\end{bmatrix}\right\}=P\begin{bmatrix}1/d & & & \\ & 1/d^2 & & \\ & & \ddots & \\ & & & 1/d^m\end{bmatrix}P^{-1}, \tag{3.a}$$

where

$$P=\begin{bmatrix}\binom{1}{1} & & & \\ \binom{2}{1} & \binom{2}{2} & & \\ \vdots & \vdots & \ddots & \\ \binom{m}{1} & \binom{m}{2} & \cdots & \binom{m}{m}\end{bmatrix}. \tag{3.b}$$

Here, m is chosen arbitrarily, and any (i, j) component of both (3.a) and (3.b) turns out to depend on i and j only. By letting $m\to\infty$, each column vector of P, which is an eigenvector of (3.a), makes us pay attention to the following sequence of numbers;

$$a^{(s)}=\{\underbrace{0, \cdots, 0}_{s-1}, \binom{s}{s}, \binom{s+1}{s}, \cdots\} \qquad (s=1, 2, \cdots). \tag{4}$$

Theorem 2. *With respect to $a^{(s)}$, it holds that $T_d[a^{(s)}]=(1/d^s)a^{(s)}(s=1, 2, \cdots)$.*

Since slight calculation verifies the equality, we omit the proof. It is to be noted that each $a^{(s)}$ is independent of the value of d. Next, we show that $\{a^{(s)}\}$ thus obtained forms a basis of L.

Lemma 3. *Let θ_s be arbitrary real numbers, and $\sum_{s=1}^{\infty}\theta_s a^{(s)}$ belongs to L. On the contrary, any element $\xi=\{\xi_1, \xi_2, \cdots\}$ is expressed as $\xi=\sum_{s=1}^{\infty}\theta_s a^{(s)}$, and the expansion coefficient θ_s is given by*

$$\theta_s=\sum_{i=1}^{s}(-1)^{s-i}\binom{s}{i}\xi_i. \tag{5}$$

Proof. The former assertion is obvious, for due to (4) each term of $\sum_{s=1}^{\infty}\theta_s a^{(s)}$ is a finite sum of real numbers. Concerning the latter one, substitute (5) into $\sum_{s=1}^{\infty}\theta_s a^{(s)}$, and we can see that its k-th term is given by

$$\sum_{s=1}^{k}\sum_{i=1}^{s}(-1)^{s-i}\binom{s}{i}\xi_i\binom{k}{s}=\sum_{i=1}^{k}\xi_i\sum_{s=i}^{k}(-1)^{s-i}\binom{s}{i}\binom{k}{s}=\sum_{i=1}^{k}\xi_i\binom{k}{i}\delta_{ki}=\xi_k.$$

Now, we are in a position to derive the spectral resolution of T_d. As is shown in the above lemma, the linear space L is a direct sum of all eigenspaces of T_d. Each eigenspace does not depend on the value of d. The projector P_s from L onto a one-dimensional subspace generated by $\{a^{(s)}\}$ is immediately obtained from (5), and we have the final result.

Theorem 4. *The summation T_d is expressed as $T_d=\sum_{s=1}^{\infty}(1/d^s)P_s$, where P_s is a projector given by*

$$(P_s[\xi])_i=\begin{cases}\sum_{j=1}^{s}(-1)^{s-j}\binom{s}{j}\xi_j\binom{i}{s} & (i\geq s),\\ 0 & (i<s).\end{cases}$$

With respect to P_s, it holds that $P_sP_t=\delta_{st}P_s$ and $\sum_{s=1}^{\infty}P_s=I$.

3. Remarks. By means of the spectral resolution of T_d, we can define a linear operator $\varphi(T_d)$ by not necessarily using the Dunford integral formalism. Here, φ is an analytic function whose pole is not equal to $1/d^s$ $(s \geq 1)$. If no zero point of φ is equal to $1/d^s$, too, the inverse of $\varphi(T_d)$ is immediately obtained, so that we can obtain the solution of the recurrence of the form $\varphi(T_d)[\xi]=u$, where ξ is unknown and u is given. This type of recurrence is treated, for example, in [2]. Also, it can be verified that T_d is a regular transformation when $d \geq 1$, while each projector P_s is neither regular nor normal.

References

[1] N. Yanagihara: Theory of Series. Asakura (1962) (in Japanese).

[2] W. Szpankowski: Solution of a linear recurrence equation arising in the analysis of some algorithms. SIAM J. Alg. Disc. Meth., **8**, 233–250 (1987).