96. On Quasi-reflexive Rings (Semigroups)

By G. W. S. van ROOYEN
Mathematics Department, University of Stellenbosch

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A right ideal I of a ring (semigroup) R is called *right quasi-reflexive* [4] if whenever A and B are right ideals of R with $AB \subseteq I$ then $BA \subseteq I$. A ring (semigroup with 0) R is said to be *right quasi-reflexive* if (0) is a right quasi-reflexive ideal of R. The concept of a left *quasi-reflexive ring* (*semigroup with* 0) is defined analogously. Evidently, semiprime rings (semiprime semigroups with 0) are left and right quasi-reflexive.

In [4] we call a ring *strongly subcommutative* if every right ideal of it is right quasi-reflexive; any ring of this class of rings is therefore right quasi-reflexive.

It is the purpose of this note to extend two results of [4], Propositions 3 and 4, and Theorem 7.4 of [2] to a wider class of rings (semigroups with 0), i.e. to the class of right quasi-reflexive rings and to the class of left and right quasi-reflexive rings (semigroups with 0), respectively. Having done that we then turn our attention to minimal (0-minimal) quasi-ideals [cf. § 2]. As a by-product, we use left and right quasi-reflexive rings to deal with a problem posed by L. Marki (cf. end of § 2).

In this note the term *ring* means associative ring (not necessarily with identity). A ring R will be called *right duo* if every right ideal is two-sided. Ideal without modifier will mean two sided ideal. A subgroup Q of $(R, +)$ is called a *quasi-ideal* of the ring R if $QR \cap RQ \subseteq Q$. A non-empty subset Q of a semigroup S is called a *quasi-ideal* of S if $QS \cap SQ \subseteq Q$. We shall call a non-zero quasi-ideal of a ring (semigroup with 0) *minimal* (*0-minimal*) if it does not properly contain any non-zero quasi-ideal. Following O. Steinfeld in [2] we say that a quasi-ideal Q of a ring R (semigroup with 0) is *canonical* if Q is the intersection of a minimal (0-minimal) right ideal K and a minimal (0-minimal) left ideal L i.e. $0 \neq Q = K \cap L$. Finally A^* will denote the right ideal $\{a - ea \mid a \in A\}$ of ring R where A is a given right ideal and e a central idempotent in R.

1. Central idempotents and right quasi-reflexive ideals.

Proposition 1 (cf. [1], Theorem 2.1). *Let R be a right quasi-reflexive ring and e an idempotent in R. The following are equivalent.*

a) *eR is a right quasi-reflexive ideal of R.*

b) *eR is an ideal R.*

c) *e is central in R.*

Proof. a)→b) is obvious. To show b)→c), note that the left annihilator of eR coincides with the right one of eR. Thence $ese = es$ for all $s \in R$.

Since $se=et$ for some $t \in R$, $es=se$ for all $s \in R$. To prove c)→a), let A and B be right ideals of R, and suppose $AB \subseteq eR$. Hence $A^*B=(0)$. Thus $BA^*=(0)$ whence $BA \subseteq eR$.

Immediately we can state

Corollary 1. *Idempotents in a right duo right quasi-reflexive ring are central.*

2. Minimal ideals and minimal (0-minimal) quasi-ideals.

Proposition 2. *Let R be a right (left) quasi-reflexive ring [semigroup with 0] and e a non-zero idempotent in R.*

The following are equivalent.

a) *eR (Re) is a minimal [0-minimal] right (left) ideal of R.*

b) *eRe is a division subring [subgroup with 0] of R [R, $\cdot$].*

c) *eRe is a minimal [0-minimal] quasi-ideal of R.*

Proof. The implication a)→b) follows from Theorem 6.11 of [2]. Assume the truth of b). Since eRe is a quasi-ideal of R, we apply Theorem 6.6 in [2]. Hence c) follows. We prove c)→a). Let $I \neq (0)$ be any right ideal of R contained in eR. Then $eI=I$. Moreover $Ie \neq (0)$. For if $Ie=(0)$, then $I \cdot eR=(0)$. Hence $eR \cdot I=(0)$. This implies $eI=(0)$ which is impossible. The minimality [0-minimality] of eRe and the fact that eIe is a non-zero quasi-ideal of R imply $I=eR$.

Corollary 2 (cf. [2], Theorem 7.4)**.** *Let R be a left and right quasi-reflexive ring (semigroup with 0) and e a non-zero idempotent in R. The following are equivalent.*

a) *Re is a minimal (0-minimal) left ideal of R.*

b) *eRe is a minimal (0-minimal) quasi-ideal of R.*

c) *eR is a minimal (0-minimal) right ideal of R.*

Corollary 3. *Let eR ($e^2=e$) be a minimal right ideal of a right quasi-reflexive ring R.*

a) *If eR is a right quasi-reflexive ideal of R, then eR is a division ring.*

b) *If the right annihilator of e in R is zero, then R is a division ring.*

Proof. a) Apply Proposition 1 to eR; hence e is central. Then use Proposition 2 to conclude the required result.

b) This part follows immediately from a) since $eR=R$.

Corollary 4(a). *Let R be a left and right quasi-reflexive ring. The following are equivalent.*

a) *R has a non-nilpotent minimal quasi-ideal.*

b) *R has a non-nilpotent minimal left ideal.*

c) *R has a non-nilpotent minimal right ideal.*

Proof. a)→b) and a)→c) follow from Theorem 3.4 in [5] and Corollary 2 above.

c)→a) (b)→a)). Let I be a non-nilpotent minimal right (left) ideal of R. Apply Proposition 6.8 in [2] to obtain $I=eR$ ($I=Re'$) for some suitable idempotent $e(e')$ in R. By Corollary 2, eRe ($e'Re'$) is a minimal quasi-ideal of R, and is clearly non-nilpotent.

Corollary 4(b). *Let S be a left and right quasi-reflexive semigroup with* 0. *The following are equivalent.*

a$'$) *S has a* 0*-minimal quasi-ideal which is not a zero semigroup.*

b$'$) *S has a* 0*-minimal left ideal containing a non-zero idempotent.*

c$'$) *S has a* 0*-minimal right ideal containing a non-zero idempotent.*

d$'$) *S has a* 0*-minimal right ideal I and a* 0*-minimal left ideal L such that* $(I\cap L)^2\neq(0)$.

Proof. a$'$)$\to$b$'$), a$'$)$\to$c$'$) and a$'$)$\to$d$'$) follow immediately from Corollary 6.4 in [2] and Corollary 2 above. b$'$)$\to$a$'$) (c$'$)$\to$a$'$)). Let L be a 0-minimal left (right) ideal containing an element $0\neq e=e^2$. So $L=Se$ ($L=eS$). By Corollary 2, eSe is a 0-minimal quasi-ideal which is not a zero semigroup. d$'$)$\to$a$'$). Since $(I\cap L)^2\neq(0)$ we apply Theorem 6.1 in [2]. Hence $I\cap L$ is a 0-minimal quasi-ideal of S which is not a zero semigroup.

Remark. Corollaries 4(a) and (b) are well known in the semiprime case. (cf. [2], Corollaries 7.5a, 7.5b). It is also well known that any minimal (0-minimal) quasi-ideal in a semiprime ring (semigroup with 0) is canonical.

We prove

Proposition 3 (cf. [3], (A)). *A minimal* (0*-minimal*) *quasi-ideal Q of a left and right quasi-reflexive ring* (*semigroup with* 0) *R is either a zero subring* (*subsemigroup*) *or a canonical quasi-ideal of R.*

Proof. Assume $Q^2\neq(0)$. Since Q is a minimal (0-minimal) quasi-ideal we apply Theorem 3.4 in [5] (Corollary 6.4 of [2]) on Q. Thus $Q=eRe=Re\cap eR$ where e is idempotent and eRe a division ring (subgroup with 0). Hence by Corollary 2, Re is a minimal (0-minimal) left ideal and eR a minimal (0-minimal) right ideal of R. Therefore Q is canonical.

The foregoing Corollary 2, Proposition 3 and Theorem 6.7 in [2] imply the following result:

C_1. *The product Q_1Q_2 of any two minimal* (0*-minimal*) *quasi-ideals Q_1 and Q_2 which are not zero subrings* (*subsemigroups*) *in a left and right quasi-reflexive ring* (*semigroup with* 0) *R is either* 0 *or a canonical minimal* (0*-minimal*) *quasi-ideal.*

This result gives a partial answer to the ring and semigroup theoretic parts of Problem 7.4 in [2] which asks if it is possible to give a wider class of rings (semigroups with 0) than that of semiprime ones, in which the assertion of the result C_2 ([2], Theorem 7.7), given below, remains valid.

C_2. *The product Q_1Q_2 of any two minimal* (0*-minimal*) *quasi-ideals Q_1 and Q_2 of a semiprime ring* (*semigroup with* 0) *R is either* 0 *or a minimal* (0*-minimal*) *quasi-ideal of R.*

In two recent articles published by O. Steinfeld (see [3]) he has proved that the assertion of the result C_2 remains true in the canonical case for any rings (semigroups with 0).

3. Example of a non-commutative left and right quasi-reflexive ring which is not semiprime:

Let $\boldsymbol{H}$ be the division ring of real quaternions and let

$$M=\begin{pmatrix} 0 & \boldsymbol{H} \\ 0 & 0 \end{pmatrix}$$

be the ring of upper triangular 2×2-matrices with zeros on the main diagonal. Clearly, M is nilpotent ($n=2$). Define R to be the direct sum of the rings M and $\boldsymbol{H}$, i.e.

$$R=M\times \boldsymbol{H}.$$

The ring R is not semiprime since the ideal $I=(M, 0)\neq 0$ of R is nilpotent.

Assume $\alpha, \beta \in R$. We show if $\alpha\beta\neq 0$ then $\beta\alpha\neq 0$.

$$\alpha\beta=\left(\begin{pmatrix} 0 & \mathrm{a} \\ 0 & 0 \end{pmatrix}, \ r_1\right)\cdot\left(\begin{pmatrix} 0 & \mathrm{b} \\ 0 & 0 \end{pmatrix}, \ r_2\right)\neq 0$$

$\Leftrightarrow r_1r_2\neq 0 \Leftrightarrow r_1\neq 0$ and $r_2\neq 0$. Hence $\beta\alpha\neq 0$. From this we conclude that R is left and right quasi-reflexive.

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