95. A Note on Isocompact wM Spaces and Mappings

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Introduction. $T_2$ isocompact $wM$ spaces behave well like $T_2$ paracompact $M$ spaces. For example, if $f: X \rightarrow Y$ is a closed, continuous map of a $T_2$ isocompact $wM$ space $X$ onto $Y$, then $Y = \bigcup_{n \geq 0} Y_n$, where, for each $n \geq 1$, $Y_n$ is discrete in $Y$ and $f^{-1}(y)$ is compact for each $y \in Y$. As such, we investigate some interesting properties of such spaces and their images under nice maps. Refer [5], [1], [4], [2] and [3] respectively, for the notions of $q$, point countable and countable type, $wM$, isocompactness, and quasi-$G_\delta$ diagonal.

Main section. Theorem 1. (i) A $T_1$ space $X$ of point countable type is a $q$ space. (ii) A regular isocompact $q$ space $X$ is point countable type.

Proof of (i). Let $x \in X$ and $K$ be a compact subset of $X$ of countable character with $x \in K$. Let $\{U_n : n \geq 1\}$ be a decreasing local base at $K$. To claim that $\{U_n\}$ is a $q$ sequence at $x$, let $x \in U_n$ for each $n$. Suppose $D = \{x_n : n \geq 1\}$ does not cluster. Then, $D = \{x_n : n \geq 1\}$ is closed. Assume $KD = \emptyset$. Then, $X - D$ is an open nhd of $K$. Since, $U_n \not\subset X - D$ for each $n$, we have a contradiction.

Proof of (ii). Let $x \in X$ and $\{U_n\}_n$ be a $q$ sequence at $x$ with $U_{n+1} \subset U_n$ for each $n$. Let $C(x) = \bigcap_n U_n$. It follows that $C(x)$ is of countable character and $x \in C(x)$. Therefore $X$ is of point countable type. Q.E.D.

Theorem 2. If a regular space $X$ with quasi-$G_\delta$ diagonal is a $q$ space or a space of point countable type, then the space is first countable.

Proof. By the Theorem 1 (i), $X$ is a $q$ space in either case. Let $\{U_n\}_n$ be a quasi-$G_\delta$ diagonal sequence. Let $x \in X$, $\{G_n\}_n$ be a $q$ sequence at $x$ and $\{n_k\}_k$ be the strictly increasing sequence of natural numbers with $x \in St(x, U_n) = \bigcup \{U \in U_n | x \in U\}$, iff $n = n_k$ for some $k \leq n$. By induction, we can obtain a sequence $\{H_m\}_m$ of open sets with $x \in H_{m+1} \subset H_m \cap G_{m+1} \cap U_{n_{m+1}}$ for each $m$, where $x \in U_{n_{m+1}}$ in $U_{n_m}$. It follows that $\{H_m : m \geq 1\}$ is a local base at $x$. Q.E.D.

Corollary 2.1. If a $T_2$ $wM$ space with quasi-$G_\delta$ diagonal is a quotient image of a locally compact, separable and metrizable space, then the space is locally compact, separable and metrizable.

Proof. Apply the Theorem 2 and a result of A. H. Stone [7]. Q.E.D.

Theorem 3. A $T_2$ isocompact $wM$ space $X$ is countable type.

Proof. Let $\{U_n\}_n$ be a decreasing $wM$ sequence and $K \subset X$ be compact. Let $W_1$ be a finite subcollection of $U_1$ with $K \subset W_1 = \bigcup W_1$. Let $W_2$ be an open collection with $K \subset \bigcup W_2$ such that $\overline{W}_2 = \{\overline{W} \mid W \in W_2\}$ refines $W_1 \wedge U_2$. Q.E.D.
Let $\mathcal{W}_2$ be a finite subcollection of $\mathcal{W}_1$ with $K \subseteq \mathcal{W}_2 = \bigcup \mathcal{W}_2$. Continuing this way, we can obtain a sequence $\{\mathcal{W}_n\}_n$ of finite open collections with $K \subseteq \mathcal{W}_n = \bigcup \mathcal{W}_n$ and $\mathcal{W}_{n+1}$ refines $\mathcal{W}_n \bigcap \mathcal{U}_{n+1}$ for each $n$. Let $D = \cap_n \mathcal{W}_n$. Then $K \subseteq D$ and $D$ is a compact set of countable character.

Q.E.D.

**Corollary 1.** A $T_3$ isocompact $wM$ space is a $k$ space.

By a result of J. E. Vaughan [8], a Tychonoff isocompact $wM$ space is a generalized $G_\delta$ set in its compactification and equivalently, its complement in its compactification is Lindelöf.

By a result of H. H. Wicke [9], a $T_3$ space is point countable type, if it is an open, continuous image of a $T_3$ isocompact $wM$ space; a $T_1$ regular isocompact space is a $q$ space, if it is an open, continuous image of a $T_2$ isocompact $wM$ space (in fact, a $T_3$ paracompact $p$ space).

**Theorem 4.** A quotient image of a regular isocompact $q$ space is a $k$ space.

**Proof.** Let $f: X \to Y$ be a quotient map of a regular isocompact $q$ space $X$ onto $Y$. Let $F \subseteq Y$ be such that $F \cap C$ is closed in $C$ for every compact $C \subseteq Y$. To claim that $F$ is closed in $Y$, we prove that $f^{-1}(F)$ is closed in $X$.

Suppose $x \in f^{-1}(F)$. For any open nhd $W$ of $y$, $f^{-1}(W) \cap C(x) \cap f^{-1}(F) \neq \emptyset$. Therefore $y \in f(C(x)) \cap F$. Since $x \in f^{-1}(F)$, we have $x \in C(x) \cap (X - f^{-1}(F))$, which implies $y \in f(C(x)) \cap (Y - F)$. Therefore $f(C(x)) \cap F$ is not closed in $f(C(x))$, which is a contradiction to the definition of $F$.

(II) Suppose $x \in C(x) \cap f^{-1}(F)$. There is an open nhd $U$ of $x$ with $U \cap C(x) \cap f^{-1}(F) = \emptyset$. Let $V_n = U \cap C(x)$ for each $n$, and $x_n \in V_n \cap f^{-1}(F)$ for each $n$. Let $x_0$ be a cluster point of the sequence $\{x_n\}_n$. Then $x_0 \in C(x) \cap U$. Let $K = \{x_n | n \geq 1\}$. Then $K$ is compact, and $x_0 \in \overline{K} \cap f^{-1}(F)$. Let $y_0 = f(x_0)$. Now $x_0 \in K$, $x_0 \in C(x) \cap U$ and $U \cap C(x) \cap f^{-1}(F) = \emptyset$ imply that $x_0 \in K \cap (X - f^{-1}(F))$. Therefore $y_0 \in f(K) \cap (Y - F)$. If $W$ is an open nhd of $y_0$, then $f^{-1}(W) \cap K \cap f^{-1}(F) \neq \emptyset$, which implies that $W \cap f(K) \cap F = \emptyset$. Therefore $y_0 \in f(K) \cap F$, which implies that $f(K) \cap F$ is not closed in $f(K)$, which contradicts the definition of $F$. Therefore $f^{-1}(F) = \overline{f^{-1}(F)}$. Q.E.D.

**Corollary 4.1.** A regular isocompact $q$ space is a $k$ space.

By a result of J. Nagata [6], we have the following corollaries.

**Corollary 4.2.** A $T_3$ space is a $k$ space, if $f$ is a quotient image of a $T_3$ isocompact $wM$ space.

**Corollary 4.3.** A $T_1$ regular isocompact $q$ space is a quotient image of a $T_3$ paracompact $M$ space.

**Theorem 5.** Let $f: X \to Y$ be a closed, continuous map of a $T_3$ isocompact $wM$ space $X$ onto $Y$. Then the following are equivalent.

(i) $Y$ is a regular $q$ space.

(ii) $Y$ is a regular space of point countable type.
(iii) The boundary $\partial f^{-1}(y)$ of $f^{-1}(y)$ is compact for each $y \in Y$.

(iv) $Y$ is a $T_\omega$ isocompact $\omega M$ space.

Proof. By the Theorems 1 and 3, we have (iv)$\Rightarrow$(ii)$\Rightarrow$(i). E. Michael has shown that (i)$\Rightarrow$(iii), [5]. We need to show, now, that (iii)$\Rightarrow$(iv): For each $y \in Y$, let

$$L(y) = \begin{cases} \partial f^{-1}(y) & \text{if } \partial f^{-1}(y) \neq \emptyset; \\ f^{-1}(y) \setminus \{p_y\}, & \text{where, } p_y \in f^{-1}(y), \text{ if } \partial f^{-1}(y) = \emptyset. \end{cases}$$

Let $X_0 = X - L$, where $L = \bigcup \{L(y)\mid y \in Y\}$. Then $X_0$ is closed in $X$, and $X_0$ is a $T_\omega$ isocompact $\omega M$ space. Let $h: X_0 \to X$ be defined by $h(x) = x$ for each $x \in X_0$. Then $g = f \circ h$ is a perfect map of $X_0$ onto $Y$. Therefore $Y$ is a $T_\omega$ isocompact (see [2]) and $\omega M$ (see [4]) space. [Note that a space being a $T_\omega$ isocompact $\omega M$ space is a perfect property.]

Q.E.D.

References