92. Note on Heinz's Inequality

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(Communicated by Kōsaku YOSIDA, M.J.A., Nov. 14, 1988)

The operator monotone functions are completely characterized by K. Löwner. But the proof is by no means short or elementary. For instance, it is not at all obvious that \( f(t) = t^{1/2} \) is operator monotone. And in fact it was discovered by E. Heinz in 1951 that \( f(t) = t \) was operator monotone for \( t \in [0, 1] \). In the following year T. Kato gave a shorter proof of the another Heinz's inequality.

In this note, it will be proved that Löwner's special case, Heinz's inequality, Heinz-Kato type inequality and the recent Chan-Kwong's result are all equivalent.

We use capital letters \( A, B, \ldots \) to denote the bounded linear operators on the Hilbert space \( \mathcal{H} \).

**Theorem.** The following results (i)–(iv) are equivalent.

(i) (K. Löwner) If \( A \geq B \geq 0 \), then \( A^{1/2} \geq B^{1/2} \).

(ii) (N. N. Chan–M. K. Kwong) If \( A \geq B \geq 0 \), \( C \geq D \geq 0 \), \( AC = CA \) and \( BD = DB \), then \( A^{1/2}C^{1/2} \geq B^{1/2}D^{1/2} \).

(iii) (E. Heinz) If \( A \geq B \geq 0 \), then \( A \geq B \) for all \( \nu \in [0, 1] \).

(iv) (E. Heinz–T. Kato) If \( A \geq B \geq 0 \), \( A \geq B \geq 0 \) for all \( \nu \in [0, 1] \).

To prove Theorem we need the following Lemmas.

**Lemma 1.** If \( A \geq B > 0 \), then \( A^{-1} \leq B^{-1} \).

**Proof.** If \( A \geq B > 0 \), then \( B^{-1/2}AB^{-1/2} \geq I \) and hence \( A^{-1} \leq B^{-1} \).

**Lemma 2.** If (i) of Theorem is fulfilled and if \( E \geq F > 0 \), \( X \geq 0 \), \( Y \geq 0 \) and \( XFX \geq Y \), then \( X \geq Y \).

**Proof.** Since \( XEX \geq XFX \geq Y \) by the assumptions, \( E^{1/2}XEXE^{1/2} \geq \) \( E^{1/2}XE^{1/2} \geq E^{1/2}YE^{1/2} \) by (i) and hence \( X \geq Y \).

**Proof of Theorem.** (i) implies (ii); For any \( \epsilon > 0 \), let \( A_\epsilon = A + \epsilon I \), then \( A_\epsilon \geq B \geq 0 \) and \( B_\epsilon^{-1} \geq A_\epsilon^{-1} \geq 0 \) by Lemma 1. Let \( X = (A_\epsilon C)^{1/2} \) and \( Y = (B_\epsilon D)^{1/2} \), then \( X \geq 0 \), \( Y \geq 0 \) and \( XA_\epsilon^{-1}X = (A_\epsilon C)^{1/2}A_\epsilon^{-1}(A_\epsilon C)^{1/2} = C \geq D = YB_\epsilon^{-1}Y \) and hence \( X \geq Y \) by Lemma 2. This implies that \( A_\epsilon^{1/2}C^{1/2} = B_\epsilon^{1/2}D^{1/2} \).

(ii) implies (iii); If \( A \geq B \geq 0 \), \( A \geq B \geq 0 \) for all \( \nu \in [0, 1] \), then \( A \geq B \) for all \( \nu \in [0, 1] \).

(iii) implies (iv); Let \( Q = V|Q| = |Q^*V \) be the polar decomposition of \( Q \), then, for any \( x, y \in \mathcal{H} \), \( ||Qx|| = ||Qx|| \leq ||Ax|| \), \( ||Q^*y|| = ||Q^*y|| \leq ||By|| \) and \( ||Q^*x|| \leq ||A^*x|| \), \( ||Q^*y|| \leq ||B^*y|| \) for all \( \nu \in [0, 1] \) by (iii) and hence \( \langle Qx, y \rangle \)
(iv) implies (i); (J. Dixmier) Since \( \| (B^{1/2})^* x \| = \| B^{1/2} x \| = \langle B x, x \rangle \leq \langle A x, x \rangle = \| A^{1/2} x \| \), let \( Q = B^{1/2}, \nu = 1/2 \) and \( x = y \) in (iv), then \( \langle B^{1/2} x, x \rangle = \| A^{1/4} x \| \leq \langle A^{1/4} x, x \rangle \) for all \( x \in \mathcal{H} \) and \( B^{1/2} \leq A^{1/2} \).

Remark. To prove the Heinz's inequality, by Theorem, we have only to show (i) is fulfilled. And it seems that the following proof is most elegant and simple (G. K. Pedersen) For any \( \varepsilon > 0 \), let \( A = A + \varepsilon I \), then \( A \geq B \geq 0 \) and \( A^{1/2} B A^{-1/2} \leq I \) and hence \( B^{1/2} A^{-1/2} \| < 1 \). Since

\[
|\sigma(A^{1/4} B^{1/2} A^{-1/4})| = \lim_{n \to \infty} \| (A^{1/4} B^{1/2} A^{-1/4})^n \|^{1/2n} = \lim_{n \to \infty} \| A^{1/4} B^{1/2} A^{-1/4} \|^{n-1} \| B^{1/2} A^{-1/4} \|^{1/n} \leq \| A_{1/4} \|^{n-1} \| (B^{1/2} A^{-1/2})^{n-1} \|^{1/(n-1)} \| B^{1/2} A^{-1/4} \|^{1/n} = |\sigma(B^{1/2} A^{-1/2})| \leq \| B^{1/2} A^{-1/2} \| < 1
\]

where \( |\sigma(A)| \) denotes the spectral radius of \( A \). \( A^{-1/4} B^{1/2} A^{-1/4} \leq I \) and \( B^{1/2} < A^{1/2} \) and hence \( B^{1/2} \leq A^{1/2} \).

References