90. Solvability in Distributions for a Class of Singular Differential Operators. II

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In [3], the author has established the local solvability in the space of distributions $\mathcal{D}'$ for some non-Fuchsian operators of hyperbolic type. In this paper, he will establish the local solvability in $\mathcal{D}'$ for a class of (non-Fuchsian) singular elliptic operators including

$$L=(t\partial_t)^2+\Delta_x+a(t,x)(t\partial_t)+\langle b(t,x),\partial_x\rangle+c(t,x).$$

As to the case of Fuchsian operators, see [2].

§1. **Theorem.** Let us consider

$$P=\sum_{j+|\alpha|\leqq m} a_{j,\alpha}(t,x)(t\partial_t)^j\partial_x^\alpha,$$

where $(t,x)=(t,x_1,\cdots,x_n)\in \boldsymbol{R}_t\times\boldsymbol{R}_x^n$, $\partial_t=\partial/\partial t$, $\partial_x=(\partial/\partial x_1,\cdots,\partial/\partial x_n)$, $m\in\{1,2,3,\cdots\}$, $\alpha=(\alpha_1,\cdots,\alpha_n)\in\{0,1,2,\cdots\}^n$, $|\alpha|=\alpha_1+\cdots+\alpha_n$ and $\partial_x^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$. On the coefficients, we assume that $a_{j,\alpha}(t,x)$ $(j+|\alpha|\leqq m)$ are C^∞ functions defined in an open neighborhood U of $(0,0)$ in $\boldsymbol{R}_t\times\boldsymbol{R}_x^n$. As to the ellipticity, we assume the following condition:

$$\sum_{j+|\alpha|=m} a_{j,\alpha}(0,0)\tau^j\xi^\alpha\neq 0, \qquad \text{when } (0,0)\neq(\tau,\xi)\in\boldsymbol{R}_\tau\times\boldsymbol{R}_\xi^n.$$

For U we write $U(+)=U\cap\{t>0\}$ and $U(-)=U\cap\{t<0\}$. Then we have

Theorem. *Let $k\in\{0,1,2,\cdots\}$. Then there is an open neighborhood U_k of $(0,0)$ in $\boldsymbol{R}_t\times\boldsymbol{R}_x^n$ which satisfies the following: for any $f(t,x)(=f)\in H^{-m-k}(U_k)$ there exists a $u(t,x)(=u)\in H^{-m-2k-1}(U_k)\cap H_{\mathrm{loc}}^{-k}(U_k(\pm))$ such that $Pu=f$ holds on U_k.*

Here, $H^{-p}(U)$ and $H_{\mathrm{loc}}^{-p}(U)$ denote the usual Sobolev spaces on U (see [1]).

Corollary. *For any $f\in\mathcal{D}'(U)$ there exists a $u\in\mathcal{D}'(U)$ such that $Pu=f$ holds in a neighborhood of $(0,0)$ in $\boldsymbol{R}_t\times\boldsymbol{R}_x^n$.*

§2. **A priori estimates.** Before giving a proof of Theorem, let us present a priori estimates for P. Put

$$P_s=\sum_{j+|\alpha|\leqq m} a_{j,\alpha}(t,x)(t\partial_t+s)^j\partial_x^\alpha.$$

Lemma. *Let P be as in §1. Then there are $\delta_k>0$ $(k=0,1,2,\cdots)$ and an open neighborhood V of $(0,0)$ in $\boldsymbol{R}_t\times\boldsymbol{R}_x^n$ such that the estimate*

$$(2.1)_k \qquad \sum_{i+|\beta|\leqq k}\|(t\partial_t+1/2)^i\partial_x^\beta(P_{1/2}\varphi)\|^2\geqq\delta_k\sum_{j+|\alpha|\leqq m+k}\|(t\partial_t+1/2)^j\partial_x^\alpha\varphi\|^2$$

holds for any $\varphi\in C_0^\infty(V(\pm))$, where $\|*\|$ means the norm in $L^2(V(\pm))$.

Proof. Note that by the change of variables $V(+)\ni(t,x)\to(\tau,x)=(-\log t,x)\in\boldsymbol{R}_\tau\times\boldsymbol{R}_x^n$ the operator P is transformed into an elliptic operator R near $(\infty,0)$. Therefore by the standard argument for elliptic operators and by using Poincaré's inequality with respect to the x-variables we can obtain

(2.2) $$\|R\psi\|^2_{(\tau,x)} \geqq \delta_0 \sum_{j+|\alpha|\leqq m} \|\partial_\tau^j \partial_x^\alpha \psi\|^2_{(\tau,x)}$$

for any $\psi \in C_0^\infty((T,\infty)\times\omega)$, if T is sufficiently large and ω is sufficiently small. Since (2.2) is equivalent to

(2.3) $$\|t^{-1/2}P\phi\|^2 \geqq \delta_0 \sum_{j+|\alpha|\leqq m} \|t^{-1/2}(t\partial_t)^j \partial_x^\alpha \phi\|^2$$

under the relation $\phi(t,x)=\psi(\tau,x)$, by putting $\varphi(t,x)=t^{-1/2}\phi(t,x)$ in (2.3) we have

$$\|P_{1/2}\varphi\|^2 \geqq \delta_0 \sum_{j+|\alpha|\leqq m} \|(t\partial_t+1/2)^j \partial_x^\alpha \varphi\|^2$$

for any $\varphi \in C_0^\infty((0,\varepsilon)\times\omega)$ (where $\varepsilon=e^{-T}$). Thus by putting $V=(-\varepsilon,\varepsilon)\times\omega$ we obtain $(2.1)_0$. The general case $(2.1)_k$ for $k\geqq 1$ can be proved inductively on k by using only the fact that $(2.1)_0$ holds for any $\varphi \in C_0^\infty(V(\pm))$.

By applying Lemma to $(P_{-m-k+1/2})^*$ (the formal adjoint operator of $P_{-m-k+1/2}$) we can obtain

Corollary to Lemma. *Let P be as in §1, and let $k \in \{0,1,2,\cdots\}$. Then there are $c_k>0$ and an open neighborhood V_k of $(0,0)$ in $\boldsymbol{R}_t \times \boldsymbol{R}_x^n$ such that the estimate*

$$\|(P_{-m-k})^*\varphi\|_k^2 \geqq c_k \|t^{m+k}\varphi\|^2_{m+k}$$

holds for any $\varphi \in C_0^\infty(V_k(\pm))$, where $\|*\|_k$ means the norm in the Sobolev space $H^k(V_k(\pm))$.

§3. Proof of Theorem. Theorem is obtained by the following three facts (A-1)–(A-3).

(A-1) Let $k \in \{0,1,2,\cdots\}$. Then there is an open neighborhood V_k of $(0,0)$ in $\boldsymbol{R}_t\times\boldsymbol{R}_x^n$ which satisfies the following: for any open subset W of V_k and any $f\in H^{-m-k}(W(\pm))$, there exists a $u\in H^{-k}(W(\pm))$ such that $P(t^{-m-k}u)=f$ holds on $W(\pm)$.

(A-2) Let $k,p\in\{0,1,2,\cdots\}$ and $u\in H^{-k}((0,T)\times\Omega)$ (where Ω is an open subset of $\boldsymbol{R}_x^n$). Then we can find a $w\in H^{-p-k}((-T,T)\times\Omega)$ such that $w=t^{-p}u$ on $(0,T)\times\Omega$ and $w=0$ on $(-T,0)\times\Omega$.

(A-3) Let $N\in\{0,1,2,\cdots\}$. Then there is an open neighborhood Ω_N of $x=0$ in $\boldsymbol{R}_x^n$ which satisfies the following: for any open subsets $\omega\Subset\omega_1$ of Ω_N and any $h\in H^{-N}((-T,T)\times\omega_1)$ satisfying $\operatorname{supp}(h)\subset\{t=0\}$, there exists a $v\in H^{-N-1+m}((-T,T)\times\omega)$ satisfying $\operatorname{supp}(v)\subset\{t=0\}$ such that $Pv=h$ holds on $(-T,T)\times\omega$.

In fact, if we know these facts, we can give a proof of Theorem as follows. Let $k\in\{0,1,2,\cdots\}$, and let $\omega\Subset\omega_1$ be sufficiently small open neighborhoods of $x=0$ in $\boldsymbol{R}_x^n$ (depending on k). Put $W=(-T,T)\times\omega$ and $W_1=(-T,T)\times\omega_1$.

Let $f\in H^{-m-k}(W)$. Choose $f_1\in H^{-m-k}(W_1)$ so that $f_1=f$ on W. Then by (A-1) we can find $u_+\in H^{-k}(W_1(+))$ and $u_-\in H^{-k}(W_1(-))$ such that $P(t^{-m-k}u_+)=f_1$ on $W_1(+)$ and $P(t^{-m-k}u_-)=f_1$ on $W_1(-)$. Moreover by (A-2) we can find a $w\in H^{-m-2k}(W_1)$ such that $w=t^{-m-k}u_+$ on $W_1(+)$ and $w=t^{-m-k}u_-$ on $W_1(-)$. Put $h=f_1-Pw$. Then we have $h\in H^{-2m-2k}(W_1)$ and $\operatorname{supp}(h)\subset\{t=0\}$. Therefore by (A-3) we have a $v\in H^{-m-2k-1}(W)$ such that

$Pv=h$ on W. Hence by putting $u=v+w$ we obtain a solution $u\in H^{-m-2k-1}(W)$ of $Pu=f$ on W. Since P is elliptic on $W(\pm)$ and since $Pu(=f)\in H^{-m-k}(W(\pm))$, the condition $u\in H^{-k}_{\text{loc}}(W(\pm))$ is clear.

Thus, to have Theorem it is sufficient to prove (A-1)–(A-3).

Proof of (A-1). Let $f\in H^{-m-k}(W(\pm))$. Put $Z=\{(P_{-m-k})^*\varphi;\ \varphi\in C_0^\infty(W(\pm))\}$ and define a linear functional T on Z by $T((P_{-m-k})^*\varphi)=\langle\varphi, t^{m+k}f\rangle$. Then by Corollary to Lemma in § 2 we can see that T is well-defined and it is continuous on Z with respect to the topology induced from $H_0^k(W(\pm))$. Therefore we can find a $u\in H^{-k}(W(\pm))$ such that $T(z)=\langle z, u\rangle$ for any $z\in Z$, that is, $\langle(P_{-m-k})^*\varphi, u\rangle=\langle\varphi, t^{m+k}f\rangle$ for any $\varphi\in C_0^\infty(W(\pm))$. Hence, we have $P_{-m-k}u=t^{m+k}f$ on $W(\pm)$ and therefore $P(t^{-m-k}u)=f$ on $W(\pm)$.

Proof of (A-2). When $k=0$, (A-2) is verified as follows: for $u\in L^2((0,T)\times\Omega)$, by defining

$$\langle w,\varphi\rangle=\Big\langle t^{-p}u, \Big(\varphi-\sum_{i=0}^{p-1}\frac{t^i}{i!}(\partial_t^i\varphi)(0,x)\Big)\Big\rangle_{L^2((0,T)\times\Omega)}$$

(for $\varphi\in C_0^\infty((-T,T)\times\Omega)$) we can obtain a $w\in H^{-p}((-T,T)\times\Omega)$ such that $w=t^{-p}u$ on $(0,T)\times\Omega$ and $w=0$ on $(-T,0)\times\Omega$.

When $k\geqq 1$, (A-2) is verified as follows. Let $u\in H^{-k}((0,T)\times\Omega)$. Then u is expressed in the form $u=\sum_{j+|\alpha|\leqq k}\partial_t^j\partial_x^\alpha(f_{j,\alpha})$ for some $f_{j,\alpha}\in L^2((0,T)\times\Omega)$. Therefore we have

$$t^{-p}u=\sum_{i+|\alpha|\leqq k}\partial_t^i\partial_x^\alpha\Big(\sum_{l=0}^{k-|\alpha|-i}t^{-p-l}g_{i,\alpha,l}\Big)\qquad\text{on } (0,T)\times\Omega$$

for some $g_{i,\alpha,l}\in L^2((0,T)\times\Omega)$. Since (A-2) with $k=0$ is already known, we can find $w_{i,\alpha,l}\in H^{-p-l}((-T,T)\times\Omega)$ such that $w_{i,\alpha,l}=t^{-p-l}g_{i,\alpha,l}$ on $(0,T)\times\Omega$ and $w_{i,\alpha,l}=0$ on $(-T,0)\times\Omega$. Hence, by putting

$$w=\sum_{i+|\alpha|\leqq k}\partial_t^i\partial_x^\alpha\Big(\sum_{l=0}^{k-|\alpha|-i}w_{i,\alpha,l}\Big)$$

we obtain a desired extension $w\in H^{-p-k}((-T,T)\times\Omega)$ in (A-2).

Proof of (A-3). Let ω_1 and h be as in (A-3). Then, by [1, Proposition 4.8 in Chapter 2] we can see that h is expressed in the form $h=\sum_{i=0}^{N-1}\delta^{(i)}(t)\otimes\mu_i(x)$ for some $\mu_i\in H^{-N+i}_{\text{loc}}(\omega_1)$ $(i=0,1,\cdots,N-1)$. Therefore, by the condition $\omega\subset\omega_1$ we have $\mu_i(=\mu_i|_\omega)\in H^{-N+i}(\omega)$ $(i=0,1,\cdots,N-1)$. Put

$$C(\rho; x,\partial_x)=\sum_{j+|\alpha|\leqq m}a_{j,\alpha}(0,x)\rho^j\partial_x^\alpha$$

and note that $C(\rho;x,\partial_x)$ is an elliptic operator near $x=0$. Put $v=\sum_{i=0}^{N-1}\delta^{(i)}(t)\otimes\psi_i(x)$. Then, we can see that $Pv=h$ is equivalent to the following recursive system of elliptic equations:

$$\text{(3.1)}\qquad\begin{cases}C(-N;x,\partial_x)\psi_{N-1}=\mu_{N-1},\\ C(-N+1;x,\partial_x)\psi_{N-2}=\mu_{N-2}+L_{N-2,N-1}(x,\partial_x)\psi_{N-1},\\ \quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\quad\vdots\\ C(-1;x,\partial_x)\psi_0=\mu_0+\sum_{l=1}^{N-1}L_{0,l}(x,\partial_x)\psi_l,\end{cases}$$

where $L_{i,l}(x,\partial_x)$ $(0\leqq i\leqq N-2$ and $i+1\leqq l\leqq N-1)$ are differential operators of order m determined by P. Therefore, if ω is sufficiently small (depending on N), we can solve (3.1) successively and obtain $\psi_i(x)\in H^{-N+i+m}(\omega)$

$(i=0,1,\cdots,N-1)$. Thus, we obtain a solution $v=\sum_{i=0}^{N-1}\delta^{(i)}(t)\otimes\psi_i(x)\in H^{-N-1+m}((-T,T)\times\omega)$ of $Pv=h$ on $(-T,T)\times\omega$.

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