86. On the Group of Units of an Abelian Extension of an Algebraic Number Field

By Yoshitaka ODAI
Gunma College of Technology

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Let K be a finite extension of the rational number field $\boldsymbol{Q}$ and L a finite abelian extension of K. For a subextension M of L/K, we denote by E_M (resp. W_M) the group of units of M (resp. the group of roots of unity in M) and define $E_{M/K}=\{\varepsilon\in E_M \mid N_{M/F}\varepsilon\in W_F$ for all subextensions $F\neq M$ of $M/K\}$, where $N_{M/F}$ is the norm map from M to F. The elements of $E_{M/K}$ are called *relative units of M over K*. We put $\mathcal{E}_M=E_{M/K}W_L/W_L\simeq E_{M/K}/W_M$. In this note we shall prove

Theorem. *Let $\mathcal{M}$ denote the set of cyclic subextensions of L/K.*

(i) $(E_L/W_L)^{[L:K]}\subseteqq\prod_{M\in\mathcal{M}}\mathcal{E}_M$ *and the product* $\prod$ *is direct.*

(ii) *Let r_1, r_2 be the numbers of real and complex places of K, respectively, and $\boldsymbol{Z}$ the ring of rational integers. For $M\in\mathcal{M}$, let r_1^M denote the number of real places of K which are unramified in M and let $\mathfrak{O}_M$ denote the ring of integers of the $[M:K]$-th cyclotomic field. Then $\mathcal{E}_M$ is an $\mathfrak{O}_M$-module. Moreover,*

$$\mathcal{E}_M\simeq\begin{cases}\boldsymbol{Z}^{r_1+r_2-1} & \text{if } M=K,\\ 0 & \text{if } M\neq K \text{ and } r_1^M+r_2=0,\\ \mathfrak{O}_M^{r_1^M+r_2-1}\oplus\mathfrak{A}_M & \text{if } M\neq K \text{ and } r_1^M+r_2>0,\end{cases}$$

where $\mathfrak{A}_M$ is a non-zero ideal of $\mathfrak{O}_M$.

This theorem has been proved in [3] and [2] if $K=\boldsymbol{Q}$, in [5] and [4] if K is an imaginary quadratic field.

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§1. **Preliminaries.** Let G be an abelian group of finite order n. Let $\boldsymbol{Q}[G]$ (resp. $\boldsymbol{Z}[G]$) denote the group ring of G over $\boldsymbol{Q}$ (resp. $\boldsymbol{Z}$). Let Λ denote the set of $\boldsymbol{Q}$-irreducible characters of G. For $\lambda\in\Lambda$, we denote $G_\lambda=\{\sigma\in G \mid \lambda(\sigma)=\lambda(1)\}$, $n_\lambda=[G:G_\lambda]$ and $\Lambda_\lambda=\{\mu\in\Lambda \mid G_\lambda\subseteqq G_\mu\}$. We define

$$e_\lambda=\frac{1}{n}\sum_{\sigma\in G}\lambda(\sigma^{-1})\sigma\in\frac{1}{n}\boldsymbol{Z}[G]\subseteqq\boldsymbol{Q}[G] \quad\text{and}\quad s_\lambda=\sum_{\sigma\in G_\lambda}\sigma\in\boldsymbol{Z}[G].$$

It is easy to see that $e_\lambda^2=e_\lambda$, $e_\lambda e_\mu=0$ $(\lambda\neq\mu)$, $\sum_{\lambda\in\Lambda}e_\lambda=1$ and

$$s_\lambda=\frac{n}{n_\lambda}\sum_{\mu\in\Lambda_\lambda}e_\mu. \tag{1}$$

Let A be a G-module. Let $\bar{A}=A/TA$, where TA is the $\boldsymbol{Z}$-torsion part of A, and let $l: A\to\bar{A}$ denote the canonical surjective G-homomorphism. We note that $\bar{A}$ can be embedded into the $\boldsymbol{Q}[G]$-module $A_{\boldsymbol{Q}}=A\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ and that $A_{\boldsymbol{Q}}=\oplus_{\lambda\in\Lambda}e_\lambda A_{\boldsymbol{Q}}$. For $\lambda\in\Lambda$, we denote $A^\lambda=\{a\in A \mid \sigma a=a$ for all $\sigma\in G_\lambda\}$; then for $a\in A^\lambda$ we have

(2) $\qquad l(a) \in \{x \in A_Q \mid \sigma x = x \text{ for all } \sigma \in G_\lambda\} = s_\lambda A_Q = \bigoplus_{\mu \in \Lambda_\lambda} e_\mu A_Q.$

Further we define

$$A_0^\lambda = \{a \in A^\lambda \mid l(a) \in e_\lambda A_Q\}.$$

Proposition. (i) *$n\bar{A} \subseteqq \sum_{\lambda \in \Lambda} l(A_0^\lambda)$ and the sum $\sum$ is direct.*

(ii) *$A_0^\lambda = \{a \in A^\lambda \mid l(s_\mu a) = 0$ for all $\mu \in \Lambda_\lambda \setminus \{\lambda\}\}$.*

(iii) *Let $\mathfrak{O}_\lambda$ denote the ring of integers of the n_λ-th cyclotomic field; then $l(A_0^\lambda)$ is an $\mathfrak{O}_\lambda$-module.*

Proof. (i) For $a \in A$, we have $na = \sum_{\lambda \in \Lambda} t_\lambda a$ where $t_\lambda = ne_\lambda \in Z[G]$. Since (1) implies $t_\lambda = n_\lambda e_\lambda s_\lambda$, we have $t_\lambda \sigma = t_\lambda$ for all $\sigma \in G_\lambda$. Therefore $t_\lambda a \in A_0^\lambda$ and $n\bar{A} \subseteqq \sum_{\lambda \in \Lambda} l(A_0^\lambda)$. As $l(A_0^\lambda) \subseteqq e_\lambda A_Q$, the sum $\sum$ is direct.

(ii) For $a \in A^\lambda$, we have from (1) and (2) that

$$l(a) \in e_\lambda A_Q \Longleftrightarrow e_\mu l(a) = 0 \text{ for all } \mu \in \Lambda_\lambda \setminus \{\lambda\}$$
$$\Longleftrightarrow l(s_\mu a) = s_\mu l(a) = 0 \text{ for all } \mu \in \Lambda_\lambda \setminus \{\lambda\}.$$

(iii) By definition $l(A_0^\lambda)$ is an $e_\lambda Z[G]$-module and we know that $e_\lambda Z[G] \simeq \mathfrak{O}_\lambda$ (cf. [2], § I, 2).

§ 2. Proof of Theorem. We take $G = \mathrm{Gal}(L/K)$ and $A = E_L$. For $\lambda \in \Lambda$, we denote by L_λ the fixed field of G_λ; then $A^\lambda = E_{L_\lambda}$. Hence (ii) of Proposition implies that $A_0^\lambda = \{\varepsilon \in E_{L_\lambda} \mid N_{L_\lambda/L_\mu}\varepsilon \in W_{L_\mu}$ for all $\mu \in \Lambda_\lambda \setminus \{\lambda\}\}$. Since $\{L_\mu \mid \mu \in \Lambda_\lambda \setminus \{\lambda\}\} = \{F \mid K \subseteqq F \subsetneqq L_\lambda\}$, we have $A_0^\lambda = E_{L_\lambda/K}$ and $l(A_0^\lambda) = \mathcal{E}_{L_\lambda}$. As $\{L_\lambda \mid \lambda \in \Lambda\} = \mathcal{M}$, (i) of Proposition proves (i) of Theorem, and (iii) of Proposition says that $\mathcal{E}_M$ is an $\mathfrak{O}_M$-module for $M \in \mathcal{M}$. Dirichlet's unit theorem says $\mathcal{E}_K \simeq Z^{r_1+r_2-1}$. Hereafter we assume $M \neq K$ and put $k = [M:K]$. Let $\varepsilon \in \mathcal{E}_M$, $\omega \in \mathfrak{O}_M$ such that $\varepsilon^\omega = 1$; then $\varepsilon^{N\omega} = 1$ where $N\omega$ is the absolute norm of ω. As $\mathcal{E}_M$ is Z-torsion free, we have $\varepsilon = 1$. It implies that $\mathcal{E}_M$ is $\mathfrak{O}_M$-torsion free. We denote by $r_{M/K}$ (resp. r_M) the Z-rank of $\mathcal{E}_M$ (resp. E_M/W_M). By (i) of Theorem we have

$$r_M = \sum_{K \subseteqq F \subseteqq M} r_{F/K} = \sum_{d \mid k} r_{M_d/K},$$

where M_d is a unique subextension of M/K of degree d. The number of real places of K which ramify in M_d is $r_1 - r_1^M$ or 0 according as k/d is odd or even. We denote by μ the Möbius's function and by φ the Euler's function; then

$$\begin{aligned} r_{M/K} = \sum_{d \mid k} \mu\left(\frac{k}{d}\right) r_{M_d} &= \sum_{\substack{d \mid k \\ k/d:\text{odd}}} \mu\left(\frac{k}{d}\right)\left\{\left(\frac{r_1 - r_1^M}{2} + r_1^M + r_2\right)d - 1\right\} \\ &+ \sum_{\substack{d \mid k \\ k/d:\text{even}}} \mu\left(\frac{k}{d}\right)\{(r_1 - r_1^M + r_1^M + r_2)d - 1\} \\ &= (r_1 - r_1^M)\left(\frac{1}{2}S_1 + S_2\right) + (r_1^M + r_2)\varphi(k), \end{aligned}$$

where

$$S_1 = \sum_{\substack{d \mid k \\ k/d:\text{odd}}} \mu\left(\frac{k}{d}\right) d \quad \text{and} \quad S_2 = \sum_{\substack{d \mid k \\ k/d:\text{even}}} \mu\left(\frac{k}{d}\right) d.$$

If k is odd then $r_1^M = r_1$, if k is even then

$$S_2 = \sum_{\substack{d \mid k \\ 2 \| k/d}} \mu\left(\frac{k}{d}\right) d = \sum_{\substack{d \mid k \\ k/d:\text{odd}}} \mu\left(\frac{2k}{d}\right) \frac{d}{2} = -\frac{1}{2} S_1.$$

Consequently

$$r_{M/K}=(r_1^M+r_2)\varphi(k).$$

On the other hand the $\boldsymbol{Z}$-ranks of $\mathfrak{O}_M$ and $\mathfrak{A}_M$ are $\varphi(k)$. Therefore Proposition 24 of [1] proves (ii) of Theorem.

References

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