85. A Cohomological Construction of Swan Representations over the Witt Ring. I

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(Communicated by Shokichi IYANAGA, M. J. A., Oct. 12, 1988)

0. Let K be a complete discrete valuation field with residue field k. We assume k is a perfect field of characteristic $p>0$. For a finite Galois extension M/K with Galois group G, the Swan character $Sw_G: G \to \mathbf{Z}$ is defined as follows.

$$Sw_G(\sigma)=\begin{cases}(1-v_M(\sigma(\pi_M)-\pi_M))\cdot f & \text{for } 1\neq\sigma\in I,\\ 0 & \text{for } \sigma\notin I,\end{cases}$$

$$Sw_G(1)=-\sum_{1\neq\sigma\in G} Sw_G(\sigma).$$

Here I denotes the inertia group, π_M a prime element of M, v_M the normalized valuation of M and f the degree of the residue field extension. Then it is a classical result that Sw_G is a character of a linear representation of G and that it can be defined over the l-adic field $\mathbf{Q}_l$ $(l\neq p)$ (resp. the fraction field of the Witt ring $W(k)$) [2], [8]. We call it the Swan representation of G and denote it by $Sw_{G,l}$ (resp. $Sw_{G,p}$).

In this note we construct $Sw_{G,p}$ cohomologically (or geometrically) *when K is of equal characteristic p*. The construction of $Sw_{G,l}$ $(l\neq p)$ was done by Katz [7]. He uses his theory of canonical extension (cf. Theorem in §3) and the machinery of l-adic etale cohomology. Instead of l-adic etale cohomology, we use a new theory of de Rham-Witt complex with logarithmic poles, which supplies us nice p-adic cohomology for open varieties. Recently, general theory of crystals with logarithmic poles has been developed independently by G. Faltings [1] and K. Kato [6].

The content of this note is as follows. In §1–2 we introduce the de Rham-Witt complex with logarithmic poles, and construct $Sw_{G,p}$ in §3. The author would like to thank Prof. K. Kato, whose observation explained in §2 is the key to the definition of de Rham-Witt complex with logarithmic poles.

1. In this and next section we introduce the de Rham-Witt complex with logarithmic poles as a preparation for §3. Here we give a short exposition concerning what is necessary in §3, and full details will be treated elsewhere. In this note we always consider sheaves and cohomologies in the etale topology.

Let k be a perfect field of characteristic $p>0$, X a smooth scheme over k and D a reduced normally crossing divisor in X. We will define sheaves of complexes $W_n\Omega^{\cdot}_X(\log D)$ (resp. $W_n\Omega^{\cdot}_X(-\log D)$), which we shall call the de Rham-Witt complex with logarithmic poles (resp. with minus logarithmic poles).

As we work with the etale topology, we may assume that there is a smooth $W=W(k)$-scheme $\mathcal{X}$ and a relatively reduced normally crossing divisor $\mathcal{D}$ such that $\mathcal{X}\otimes_W k\simeq X$ and $\mathcal{D}\otimes_{\mathcal{X}} X=D$. Let $D=\sum D_i$ (each D_i is irreducible). Then $\mathcal{D}=\sum \mathcal{D}_i$ where $D_i=\mathcal{D}_i\otimes_W k$. We may assume moreover that there is a "frobenius" $f:\mathcal{X}\to\mathcal{X}$ such that f induces the absolute frobenius on X and $f^*\mathcal{D}=p\cdot\mathcal{D}$. For simplicity, we denote $\mathcal{X}_n=\mathcal{X}\otimes_W W_n$ where $W_n=W_n(k)$. Consider the de Rham complex with logarithmic poles

$DR_{\mathcal{X}_n}(\log\mathcal{D}):\ \mathcal{O}_{\mathcal{X}_n}\longrightarrow\Omega^1_{\mathcal{X}_n}(\log\mathcal{D})\longrightarrow\Omega^2_{\mathcal{X}_n}(\log\mathcal{D})\longrightarrow\cdots$

$DR_{\mathcal{X}_n}(-\log\mathcal{D}):\ \mathcal{J}\otimes\mathcal{O}_{\mathcal{X}_n}\longrightarrow\mathcal{J}\otimes\Omega^1_{\mathcal{X}_n}(\log\mathcal{D})\longrightarrow\mathcal{J}\otimes\Omega^2_{\mathcal{X}_n}(\log\mathcal{D})\longrightarrow\cdots,$

where $\Omega^i_{\mathcal{X}_n}(\log\mathcal{D})$ is the differential forms with logarithmic poles along $\mathcal{D}\otimes_W W_n$ and $\mathcal{J}$ denotes the ideal sheaf of $\mathcal{D}$.

The key point is the observation due to K. Kato that the above complex does not depend on the choice of $\mathcal{X}$, $\mathcal{D}$ and f in the derived category. This point will be explained in §2.

Now we can define the de Rham-Witt complex with logarithmic poles by the method of Illusie-Raynaud [5] III (1.5). We define

$$W_n\Omega^i_X(\pm\log D):=\mathcal{H}^i(DR_{\mathcal{X}_n}(\pm\log\mathcal{D})).$$

These are naturally considered as coherent $W_n(\mathcal{O}_X)$-modules. The boundary $d:\ W_n\Omega^i_X(\pm\log D)\longrightarrow W_n\Omega^{i+1}_X(\pm\log D)$ is defined to be the boundary map induced from the exact sequence

$$0\longrightarrow DR_{\mathcal{X}_n}(\pm\log\mathcal{D})\longrightarrow DR_{\mathcal{X}_{2n}}(\pm\log\mathcal{D})\longrightarrow DR_{\mathcal{X}_n}(\pm\log\mathcal{D})\longrightarrow 0.$$

We next define the restriction $\pi:\ W_{n+1}\Omega^i_X(\pm\log D)\longrightarrow W_n\Omega^i_X(\pm\log D)$. One checks that the endomorphism (f/p^{i-1}) on $\Omega^i_{\mathcal{X}/W}(\log\mathcal{D})$ (resp. $\mathcal{J}\otimes\Omega^i_{\mathcal{X}/W}(\log D)$) induces an injection

$$p:\ W_n\Omega^i_X(\pm\log D)\longrightarrow W_{n+1}\Omega^i_X(\pm\log D)$$

whose image coincides with $p\cdot W_{n+1}\Omega^i_X(\log D)$. This definition is independent of the choice of f, as is seen from the product construction in §2. Then we define π to be the surjective homomorphism which makes the following diagram commutative.

$$\begin{array}{ccc} W_{n+1}\Omega^i_X(\pm\log D) & \xrightarrow{\pi} & W_n\Omega^i_X(\pm\log D) \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle p} \\ & & W_{n+1}\Omega^i_X(\pm\log D). \end{array}$$

We define $W\Omega^\cdot_X(\pm\log D)=\varprojlim_\pi W_n\Omega^\cdot_X(\pm\log D)$. The operator

$$F:\ W_{n+1}\Omega^i_X(\pm\log D)\longrightarrow W_n\Omega^i_X(\pm\log D)$$

$$(\text{resp. } V:\ W_n\Omega^i_X(\pm\log D)\longrightarrow W_{n+1}\Omega^i_X(\pm\log D))$$

is defined to be the map induced from the natural projection

$$DR_{\mathcal{X}_{n+1}}(\pm\log\mathcal{D})\longrightarrow DR_{\mathcal{X}_n}(\pm\log\mathcal{D})$$

$$(\text{resp. "}p\text{"}:\ DR_{\mathcal{X}_n}(\pm\log\mathcal{D})\longrightarrow DR_{\mathcal{X}_{n+1}}(\pm\log\mathcal{D})).$$

There is a relation between the de Rham-Witt complex with logarithmic poles and the de Rham-Witt complex. Here we restrict ourselves to the case $\dim X=1$, as it suffices for later use. Let X be a proper smooth curve over k, and D_0 (resp. D_∞) be a disjoint union of closed points of X. We assume $D_0\cap D_\infty=\phi$ and define

$$W_n\Omega^{\cdot}_X(\log D_0-\log D_\infty) \quad \text{and} \quad W\Omega^{\cdot}_X(\log D_0-\log D_\infty)$$

to be the de Rham-Witt complex with logarithmic poles along D_0 and with minus logarithmic poles along D_∞. As is seen from the construction, we have exact sequences of complexes

$$(*) \quad \begin{aligned} &0\longrightarrow W_n\Omega^{\cdot}_X(\log D_0-\log D_\infty)\longrightarrow W_n\Omega^{\cdot}_X(\log D_0)\longrightarrow i_\infty * W_n\Omega^{\cdot}_{D_\infty}\longrightarrow 0,\\ &0\longrightarrow W_n\Omega^{\cdot}_X\longrightarrow W_n\Omega^{\cdot}_X(\log D_0)\longrightarrow i_0 * W_n\Omega^{\cdot}_{D_0}[-1]\longrightarrow 0,\end{aligned}$$

where i_0 (resp. i_∞) denotes the closed immersion, and $[-1]$ denotes the shift of the complex. By passing to the limit, we obtain

$$(**) \quad \begin{aligned} &0\longrightarrow W\Omega^{\cdot}_X(\log D_0-\log D_\infty)\longrightarrow W\Omega^{\cdot}_X(\log D_0)\longrightarrow i_\infty * W\Omega^{\cdot}_{D_\infty}\longrightarrow 0,\\ &0\longrightarrow W\Omega^{\cdot}_X\longrightarrow W\Omega^{\cdot}_X(\log D_0)\longrightarrow i_0 * W\Omega^{\cdot}_{D_0}[-1]\longrightarrow 0.\end{aligned}$$

Lemma. (1) *$H^q(X, W\Omega^{\cdot}_X(\log D_0-\log D_\infty))$ is a free W-module of finite rank for all $q\geq 0$ and vanishes for all $q\geq 3$.*

(2) *If $D_0\neq\phi$, we have* $H^2(X, W\Omega^{\cdot}_X(\log D_0-\log D_\infty))=0$.

(3) *If $D_\infty\neq\phi$, we have* $H^0(X, W\Omega^{\cdot}_X(\log D_0-\log D_\infty))=0$.

By $(*)$, each $H^q(X, W_n\Omega^{\cdot}_X(\log D_0-\log D_\infty))$ is a finitely generated W-module. So

$$H^q(X, W\Omega^{\cdot}_X(\log D_0-\log D_\infty))=\varprojlim_{n} H^q(X, W_n\Omega^{\cdot}_X(\log D_0-\log D_\infty)).$$

By definition, (2) (resp. (3)) is reduced to the fact $H^1(X, \Omega^1_X(\log D_0))=0$ (resp. $H^0(X, \mathcal{J}_{D_\infty})=0$). The assertion (1) can be seen from the assumption $\dim X=1$.

2. In this section we explain how one sees that $DR_{\mathcal{X}_n}(\pm\log\mathcal{D})$ defined in §1 does not depend on the choice of liftings $\mathcal{X}, \mathcal{D}$ and f in the derived category.

Choose another lifting $\mathcal{X}', \mathcal{D}'$ and f'. Let $h: \mathcal{X}\to\mathcal{X}\otimes_W\mathcal{D}'$ be the blowing up defined by the product ideal of the ideals defined by $\mathcal{D}_i\times_W\mathcal{D}'_i$ $(1\leq i\leq a)$, and let $\mathcal{U}$ be the complement of the strict transforms of the closed subschemes $\mathcal{X}\times_W\mathcal{D}'_i$ and $\mathcal{D}_i\times_W\mathcal{X}'$ $(1\leq i\leq a)$. By direct calculation, we see that $\mathcal{U}\to\mathcal{X}$ (resp. $\mathcal{U}\to\mathcal{X}'$) is smooth, and that the inverse image $\tilde{\mathcal{D}}$ in $\mathcal{U}$ of $\mathcal{D}\times_W\mathcal{D}'$ coincides with the inverse image of $\mathcal{D}$ (resp. $\mathcal{D}'$). Moreover there is a closed immersion $X\hookrightarrow\mathcal{U}$ such that $X\hookrightarrow\mathcal{U}\to\mathcal{X}\times_W\mathcal{X}'$ coincides with the diagonal embedding. For this, note that the locus of the blowing-up is codimension one in X.

Let $\mathcal{P}$ be the structure sheaf of the divided power envelope of $\mathcal{U}_n=\mathcal{U}\otimes_W W_n$ with respect to the ideal defined by the image of X. We define complexes

$$\mathcal{P}\otimes DR_{\mathcal{U}_n}(\log\tilde{\mathcal{D}}): \mathcal{P}\longrightarrow\mathcal{P}\otimes_{\mathcal{O}}\Omega^1_{\mathcal{U}_n}(\log\tilde{\mathcal{D}})\longrightarrow\mathcal{P}\otimes_{\mathcal{O}}\Omega^2_{\mathcal{U}_n}(\log\tilde{\mathcal{D}})\longrightarrow,$$

$$\mathcal{P}\otimes DR_{\mathcal{U}_n}(-\log\tilde{\mathcal{D}}): \mathcal{J}\mathcal{P}\longrightarrow\mathcal{J}\mathcal{P}\otimes_{\mathcal{O}}\Omega^1_{\mathcal{U}_n}(\log\tilde{\mathcal{D}})\longrightarrow\mathcal{J}\mathcal{P}\otimes_{\mathcal{O}}\Omega^2_{\mathcal{U}_n}(\log\tilde{\mathcal{D}})\longrightarrow,$$

where $\mathcal{O}=\mathcal{O}_{\mathcal{U}_n}$ and $\mathcal{J}$ denotes the ideal sheaf of $\tilde{\mathcal{D}}$.

Now the following lemma shows $DR_{\mathcal{X}_n}(\pm\log\mathcal{D}_n)\simeq DR_{\mathcal{X}'_n}(\pm\log\mathcal{D}'_n)$.

Lemma. *The natural homomorphisms*

$$DR_{\mathcal{X}_n}(\pm\log\mathcal{D})\longrightarrow\mathcal{P}\otimes DR_{\mathcal{U}_n}(\pm\log\tilde{\mathcal{D}}) \quad \text{and}$$

$$DR_{\mathcal{X}'_n}(\pm\log\mathcal{D}')\longrightarrow\mathcal{P}\otimes DR_{\mathcal{U}_n}(\pm\log\tilde{\mathcal{D}})$$

are quasi-isomorphisms.

We give a proof for the first morphism. The problem is etale local on X. So we may assume $\mathcal{U}=\mathcal{X}\otimes_W W[S_1, \cdots, S_d]$. Hence we have

$$\mathcal{P}\otimes DR_{\mathcal{U}_n}(\log \tilde{\mathcal{D}})\simeq DR_{\mathcal{X}_n}(\pm\log \mathcal{D}_n)\otimes_{W_n}(\Omega^{\cdot}_{W_n[S]}\otimes_{W_n[S]} W_n\langle S\rangle),$$

where $W_n[S]$ (resp. $W_n\langle S\rangle$) denotes the polynomial ring (resp. divided power algebra) in d variables $S_1, \cdots, S_d$. The lemma follows from the fact that $\Omega^{\cdot}_{W_n[S]}\otimes_{W_n[S]} W_n\langle S\rangle$ is quasi-isomorphic to W_n.

(to be continued.)

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